

Weighted(k, n) – arcs of Type($n - 11, n$) in $PG(2, 11)$

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Abstract

In this paper, we discussed a weighted (k, n) –arcs of type $(n - 11, n)$ in projective plane of order eleven. We found a new methodology for the construction of the weighted (k, n) –arcs of two types. We proved that there exist a $(78, n; f)$ –arcs of type $(n - 11, n)$ when the points of weight zero is fifty five and the points of weight one form an envelope. Finally, we proved that there is a weighted (k, n) –arcs of type $(n - 11, n)$ when the points of weight one lies on only one n –weighting line of a weighted (k, n) –arcs and the points of weight two lies on exactly two n –weighting lines.

Key words: Projective plane of order 11, Weighted (k, n) –arcs of two types.

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1. Introduction

The notion of weighted (k, n, f) – arcs were proposed by Scafati (1971), Barnabei (1979) studied these types of arcs and obtained some particular results about the existence and non-existence of these arcs in $PG(2, q)$ by using a computer programmer. D'Agostini (1979) studied caps with weighted points in $PG(2, q)$, some relations between parameters of (k, n, f) –caps and characters were found. In particular, D'Agostini (1980) studied a weighted (k, n) –arcs of type $(n - 2, n)$ in $PG(2, q)$. Wilson (1986) proved that there is $(88, 14, f)$ –arc of type $(11, 14)$ in the Galois plane of order 9. Also he proved that there is $(10, 7, f)$ –arc of type $(4, 7)$ in $PG(2, 3)$. Hameed (1989) studied the existence and non-existence weighted (k, n) –arcs in $PG(2, 9)$ and he proved that there exist $(81, 12, f)$ –arc of type $(9, 12)$ and $(85, 13, f)$ –arc of type $(10, 13)$. Mahmood (1990) discussed (k, n, f) – arcs in $PG(2, 5)$. She proved there is $(21, 11, f)$ –arc of type $(6, 11)$, $(20, 10, f)$ –arc of type $(5, 10)$ and $(15, 8, f)$ –arc of type $(3, 8)$. The extensions work of the weighted arcs in $PG(2, 9)$ was investigated by Abass (2011). He proved there exist $(81, 12, f)$ –arc

of type $(9, 12)$ and $(76, 11, f)$ –arc of type $(8, 11)$. Abudljebar (2012), studied the generalization of (k, n, f) – arcs of type $(1, n)$ in $PG(2, q)$ and weighted(k, n) –arcs of type $(n - 4, n)$ in $PG(2, 8)$.

Most of the authors were used the table of lines and coordinates of points, to construct the example of a weighted(k, n) –arcs in $PG(2, q)$, while we used a new technique depending on the dual of k -arcs and the envelope in projective plane of order eleven.

2. Preliminaries

Definition 2.1 [Hirschfeld]. Let $GF(p) = \mathbb{Z}/p\mathbb{Z}$, p is prime and let $f(x)$ be irreducible polynomial of degree h over $GF(p)$, then

$$GF(q) = GF(p^h) = \frac{GF(p)[x]}{f(x)} = \{a_0 + a_1t + \dots + a_{h-1}t^{h-1} : a_i \text{ in } GF(p), f(t) = 0\}$$

Definition 2.2 [Hirschfeld]. A projective plane over $GF(q)$ is 2-dimentional projective space and denoted by $PG(2, q)$ or π which contains $q^2 + q + 1$ lines, every line contains $q + 1$ points and satisfy the following axioms:

- (i) Any two distinct points determine a unique line;
- (ii) Any two distinct lines intersect in exactly one point;
- (iii) There exist four distinct points such that no three of them are collinear.

Definition 2.3 [Hirschfeld]. A t_n -arc is a set of t_n points such that no three points are collinear.

Lemma 2.4 [Hirschfeld]. Let $t(p)$ be the number of tangents through p of t_n –arc, T_i be the number of i –secants of t_n in $PG(2, q)$, then

- (i) $t(p) = q + 2 - t_n$;
- (ii) $T_2 = \frac{t_n(t_n-1)}{2}$;
- (iii) $T_1 = t_n t$, $t = q + 2 - t_n$;
- (iv) $T_0 = \frac{q(q-1)}{2} + \frac{t(t-1)}{2}$;
- (v) $T_0 + T_1 + T_2 = q^2 + q + 1$.

Definition 2.5[Hirschfeld]. The set of t_n lines such that no three are concurrent is called a dual of t_n -arc.

Definition 2.6 [Hirschfeld]. The set of $(q + 1)$ lines such that no three of them are concurrent is called an envelope and which is represent a dual of the conic in $PG(2, q)$, when q is odd.

Lemma 2.7. Let $t(l)$ be the number of points lies on l and let S_i be the number of points which pass through it i 2-secant, then

- (i) $t(l) = q + 2 - t_n$;
- (ii) $S_2 = \frac{t_n(t_n-1)}{2}$;

$$(iii) S_1 = t_n t, \quad t = q + 2 - t_n;$$

$$(iv) S_0 = \frac{q(q-1)}{2} + \frac{t(t-1)}{2};$$

$$(v) S_0 + S_1 + S_2 = q^2 + q + 1.$$

Definition 2.8 [Hirschfeld]. A (k, n) -arc $\mathcal{H}$ is a set of k points such that there are n but no $n + 1$ them are collinear.

Lemma 2.9 [Wilson]. For the (k, n) -arc $\mathcal{H}$, the following equations are hold:

$$(i) \sum_{i=0}^n \tau_i = q^2 + q + 1;$$

$$(ii) \sum_{i=1}^n i \tau_i = k(q + 1);$$

$$(iii) \sum_{i=2}^n \frac{i(i-1)}{2} \tau_i = \frac{k(k-1)}{2},$$

where τ_i are the number of i -secants of (k, n) -arc such that $\mathcal{H} \cap \tau = i$.

Definition 2.10 [D'Agostini]. Let π be the projective plane of order q and denoted by p and R are respectively the sets of points and lines of π . Let f be a function from P into the set N of non-negative integers and call the weight of $p \in P$ the value $f(p)$ and the support of f the set of points of the plane have non-zero weight. By using f we can define a function $F: R \rightarrow N$ such that for any $r \in R$, $F(r) = \sum_{p \in r} f(p)$. We call $F(r)$ the weight of the line r .

Definition 2.11 [D'Agostini]. A $(k, n; f)$ -arc of the plane π is a subset K of the points of the plane such that

$$(i) K \text{ is the support of } f;$$

$$(ii) k = |K|;$$

$$(iii) n = \max\{F(r): r \in R\}.$$

Denote $\omega = \max_{p \in P} f(p)$, V_i^j to the number of lines of weight i through a point of weight j and $W = \sum_{j=0}^{\omega} \mathcal{H}_j = \sum_{p \in P} f(p)$. For a $(k, n; f)$ -arc, we have the following important Lemma:

Lemma 2.12 [Hameed]. For the weighted (k, n) -arcs in $PG(2, 11)$, the following statements are holds:

$$(i) \omega \leq 11;$$

$$(ii) \text{ If } p \text{ is any point of the plane, then } \sum_{r \in [p]} F(r) = W + qf(p), \text{ where } [p] \text{ denote the set of lines through } p;$$

$$(iii) \text{ The weight } W \text{ of a weighted } (k, n) \text{-arc satisfies } (n - 11)(q + 1) \leq W \leq (n - \omega)q + n;$$

$$(iv) \text{ Let } K \text{ be a weighted } (k, n) \text{-arc of type } (n - 11, n), n - 11 > 0 \text{ and let } p \text{ be a point having weight } s, \text{ then } V_m^s \text{ and } V_n^s \text{ are determined } p \text{ and are given by:}$$

$$V_{n-11}^s = \frac{q(n - s) - W + n}{11}$$

and

$$V_n^s = \frac{q(s - n + 11) + W - n + 11}{11};$$

(v) $q \equiv 0 \pmod{11}$;

(vi) $k = \sum_{j=1}^2 l_j$;

(vii) The characters of a weighted $(k, n) - \text{arcs } K$ of type $(n - 11, n)$ are given by

$$t_{n-11} = \left[\frac{q+1}{11} \right] \left[\frac{n(q^2 + q + 1)}{q+1} - W \right]$$

and

$$t_n = \left[\frac{q+1}{11} \right] \left[W - \frac{(n-11)(q^2 + q + 1)}{q+1} \right]$$

Corollary 2.13 [Hameed]. If $W = (n - 11)(q + 1)$, then a weighted $(k, n) - \text{arc}$ is minimal and if $W = (n - \omega) + n$, then a weighted $(k, n) - \text{arc}$ is maximal.

Principle of Duality 2.14 [Hirschfeld]. For any space $S = PG(n, q)$, there is a dual space S^* , whose points and primes are respectively primes and points of S . For any theorem true in $\square$, there is an equivalent theorem true in $\square^*$.

3. weighted $(\square, \square) - \text{arcs of type } (\square - 11, \square)$

For a weighted $(\square, \square) - \text{arcs of type } (\square - 11, \square)$, it is necessary that $\square \geq 11$. If $\square = 11$ we have to consider a $(\square, 11) - \text{arc}$ having only 0 –weighting line and 11 –weighting lines, this arc is maximal.

Lemma 3.1. The existence of a weighted $(\square, \square) - \text{arcs of type } (\square - 11, \square)$ in $\square\square(2, \square)$ with $\square + 1 < \square < 2\square + 2$ requires $\square \equiv 0 \pmod{11}$

Proof. Directly, from Lemma 2.12 case (v). ■

Lemma 3.2 [D'Agostini]. The existence of a weighted $(\square, \square) - \text{arcs of type } (\square - 11, \square)$ in $\square\square(2, \square)$ with $\square + 1 < \square < 2\square + 2$ requires $\square_{\square} = 0, \square \geq 3$.

We used Lemma 2.12 case (iii) to get

$$(\square - 11)(\square + 1) \leq \square \leq (\square - 11)(\square + 1) + 11$$

Lemma 3.3. For a weighted $(\square, \square) - \text{arcs of type } (\square - 11, \square)$ in $\square\square(2, \square)$ with $\square$ minimal ($\square = (\square + 1)(\square - 11)$) we have

$$\begin{aligned} \square_{\square-11}^0 &= \square + 1, & \square_{\square-11}^1 &= \frac{10\square + 11}{11}, & \square_{\square-11}^2 &= \frac{9\square + 11}{11} \\ \square_{\square}^0 &= 0, & \square_{\square}^1 &= \frac{\square}{11}, & \square_{\square}^2 &= \frac{2\square}{11} \end{aligned}$$

Proof. From Lemma 2.12 case (iv), by substituting $\square - 11 = \square + 1$ for $\text{Im } \square = \{0, 1, 2\}$. ■

Corollary 3.4. There is no points of weight 0 on $\square -$ weighting lines of a weighted $(\square, \square) -$ arcs of type $(\square - 11, \square)$.

For the case $\square_0 > 0, \square_1 > 0, \square_2 > 0$ and $\square_\square = 0$ where $3 \leq \square \leq 11$, we have the weight of the points of the $(\square, \square; \square) -$ arc is $\omega = 2$, and by using the minimal case $(\square = (\square - 11)(\square + 1))$ and by the counting the number of lines of $\square\square(2, \square)$ we find the following:

$$\square_\square + \square_{\square-11} = \square^2 + \square + 1$$

By counting the number of $\square -$ weighting lines $\square_\square$ and $(\square - 11) -$ weighting lines $\square_{\square-11}$, and counting the total incidence, it follows that

$$\square\square_\square + (\square - 11)\square_{\square-11} = \square(\square + 1) = (\square - 11)(\square + 1)^2$$

Consequently, we get

$$\square_\square = \frac{(\square-11)\square}{11} \quad (3.1)$$

$$\square_{\square-11} = \frac{11\square^2 + (22-\square)\square + 11}{11} \quad (3.2)$$

Lemma 3.5. The $\square -$ weighting lines of a weighted $(\square, \square) -$ arcs of type $(\square - 11, \square)$ form a dual of $\square_\square -$ arc in $\square\square(2, 11)$.

Proof. From Lemma 3.3, we have $\square_\square^2 = 2$, this mean that there are no three $\square -$ weighting lines are concurrent. Then the number of $\square -$ weighting lines $\square_\square$ form a dual of $\square_\square -$ arc. ■

Suppose that on $\square -$ weighting lines there are α points of weight 1 and β points of weight 2. Then counting the points of $\square -$ weighting lines, it follows that:

$$\alpha + \beta = \square + 1$$

And counting the weight of points on $\square -$ weighting lines, we have

$$\alpha + 2\beta = \square$$

Solving these two equations, we obtain

$$\alpha = 2(\square + 1) - \square \quad (3.3)$$

$$\beta = \square - (\square + 1) \quad (3.4)$$

counting the incidences between the points of weight 2 and $\square -$ weighting lines, we get

$$\square_2\square_\square^2 = \square_\square\square$$

Making use of Lemma 3.3, equation (3.1) and equation (3.4) we obtain

$$\square_2 = \frac{(\square-11)(\square-\square-1)}{2} \quad (3.5)$$

Similarly, counting the incidences between the points of weight l and $\square$ –weighting lines, we have

$$\square_l \square_{\square}^l = \square_{\square} \square$$

Hence, by using Lemma 3.3, equation (3.2) and equation (3.3), we get

$$\square_l = (\square - 11)(2\square + 2 - \square) \quad (3.6)$$

From equations (3.5) and (3.6), counting the points in the plane

$$\begin{aligned} \square_0 + \square_l + \square_2 &= \square^2 + \square + 1 \\ \square_0 &= \square^2 + \square + 1 - (\square - 11)(2\square + 2 - \square) - \frac{(\square - 11)(\square - \square - 1)}{2} \end{aligned}$$

Hence

$$2\square^2 + (35 - 3\square)\square + \square^2 - 14\square + 35 - 2\square_0 = 0 \quad (3.7)$$

The solution of equation (3.7) exists with respect to $\square$ if $(35 - 3\square)^2 - 8(\square^2 - 14\square + 35 - 2\square_0)$ is square, then

$$(\square - 49)^2 - (1456 - 16\square_0) = \square\square\square\square\square \quad (3.8)$$

We discuss the $(\square, \square; \square)$ –arcs of type $(\square - 11, \square)$ in $\square\square(2, 11)$ when the points of weight 0 equal to 55. For the value of $\square_0 = 55$, the equation (3.7) becomes

$$2\square^2 + (35 - 3\square)\square + \square^2 - 14\square - 75 = 0 \quad (3.9)$$

From the equation (3.8), we conclude that $((\square - 49)^2 - 576)$ must be square.

For the value of $\square = 11$, the equation (3.9) becomes

$$\square^2 - 47\square + 552 = 0$$

We have two solutions for $\square$ being non-negative with $\square = 0\square\square\square(\square - (\square - 11))$ which are $\square = 24$ or $\square = 23$.

For $n=24$, the weight of the lines either $2\square + 2$ or zero and this gives underling $(\square, \square)$ –arcs in $\square\square(2, \square)$

4. $(\square\square, \square\square; \square)$ –arcs of Type $(\square\square, \square\square)$ in $\square\square(\square, \square\square)$

Lemma 4.1. For a $(78, 23; \square)$ –arc of type $(12, 23)$ in $\square\square(2, 11)$ for the minimal $\square = 144$, we have

$$\square_{12}^0 = 12, \square_{12}^l = 11, \square_{12}^2 = 10, \square_{23}^0 = 0, \square_{23}^l = 1, \square_{23}^2 = 2$$

Proof. From Lemma 3.3 directly, we get the requirements by putting $\square = 23, \square = 11$. $\square$

Lemma 4.2. For the existence of a minimal $(78, 23; \square)$ –arc of type $(12, 23)$ with $\square_0 = 55$ in $\square\square(2, 11)$ we have:

- (i) The number of 23 –weighting lines $(\square_{23})$ is 12;
- (ii) The number of 12 –weighting lines $(\square_{12})$ is 121;
- (iii) The number of points of weight 2 $(\square_2)$ is 66;
- (iv) The number of points of weight 1 $(\square_1)$ is 12.

Proof. From equations (3.1), (3.2), (3.5) and (3.6) we obtain (i), (ii), (iii) and (iv) respectively.

Let $\square$ be a 12-weighting line having on it $\square$ points of weight 2, $\square$ points of weight 1 and $\square$ points of weight 0. Then, counting points on $\square$ gives

$$\square + \square + \square = \square + 1 = 12 \quad (4.1)$$

and summing the weights of points on $\square$ gives

$$2\square + \square = \square - 11 = 12 \quad (4.2)$$

So the possible non-negative integers solutions of the above equations are 7.

Lemma 4.3. (i) The points of weight two form a $(66, 11)$ –arc of type $(12, 0, 0, 0, 0, 55, 66, 0, 0, 0, 0, 0)$.

(ii) The points of weight one form a $(12, 12)$ –arc of type $(0, 0, 0, 0, 0, 0, 0, 0, 0, 66, 12, 55)$.

(iii) The points of weight zero form a $(55, 6)$ –arc of type $(55, 66, 0, 0, 0, 0, 12)$.

Proof. Directly, from the equations (3.3) and (3.4).

For (ii) and (iii), since $\square \leq 6$, $\square \leq 12$, and $\square \leq 6$, then we get the requirements. $\square$

Remark. (i) From Lemma 2.7, the number of points which pass through their two $\square$ -weighting lines is 66 and equal to the number of points of weight 2.

(ii) The number of points which pass through their one $\square$ -weighting line is 12 and equal to the number of points of weight 1.

From Lemma 2.12, we have $\square = \sum_{\square=1}^2 \square_{\square} = 78$.

Hence we deduce the following theorem.

Theorem 4.4. There exist a $(78, 23; \square)$ –arc of type $(12, 23)$ in $\square\square(2, 11)$ when the points of weight 0 is 55 and the points of weight 1 form an envelope. $\square$

For the case $\square_{\theta} = 56, 58, 61$ and 65

From equation 3.7, by putting $\square_{\theta} = 56, 58, 61$ and 65 we have $\square = 22, 21, 20$ and 19 respectively.

From Lemma 2.12, $\square = 22, 21, 20$ and 19 we have the minimal case $W=132, 120, 108$ and 96 respectively.

From Lemma 3.3, we have $\square_{\square}^2 = 2$, this mean that the number of $\square$ -weighting lines form a dual $\square_{\square}$ –arc and the remaining points of $\square\square(2, 11)$ are classified to the following types:

(i) The points which pass through their two $\square$ -weighting line and are represent points of weight two which equal to $\frac{\square_{\square}(\square_{\square}-1)}{11}$.

(ii) The points which pass through their one $\square$ -weighting line and are represent points of weight one which equal to $\square_{\square}(\square + 2 - \square_{\square})$.

Theorem 4.5. There exist a $(\square, \square; \square)$ -arc, of type $(\square - 11, \square)$ in $\square\square(2, 11)$ when the points of weight one lies on only one $\square$ –weighting line of $(\square, \square; \square)$ -arc and the points of weight two lies on exactly two $\square$ –weighting lines and the points of weight zero equal 56, 58, 61 and 65.

Proof:

Let the number of points of weight zero equal 56, from Lemma 2. 7, the number of points of weight one equal 22 and the points of weight two equal 55. And the points of weight two form a $(55, 10)$ –arc, the points of weight one form a $(22, 11)$ –arc and the points of weight zero form a $(56, 6)$ –arc.

$$\text{Hence } \square = \sum_{\square=1}^2 \square_{\square} = 77.$$

By the same arguments, for the number of points of weight 0 equal 58, 61 and 65, the number of points of weight 1 equal 30, 36 and 28 and the number of points of weight 2 equal 45, 36 and 40, then we get $\square = 75, 72$ and 68 respectively. $\square$

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الأقواس الموزونة- (k, n) من النوع $(n-11, n)$ في المستوي الأسقاطي من الرتبة 11

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الخلاصة

في هذا البحث برهنا وجود الأقواس الموزونة من النوع $(n-11, n)$ في المستوي الأسقاطي من الرتبة 11. وجدنا منهجية جديدة لبناء الأقواس الموزونة من النوعين. لقد ثبتنا وجود القوس $(78, n; \square)$ من النوع $(n-11, n)$ عندما عدد النقاط التي لها الوزن صفر تساوي (55)، والنقاط التي لها وزن واحد هي $envelope(12)$. وأخيراً، ثبتنا وجود الأقواس الموزونة من النوع $(n-11, n)$ عندما عدد النقاط من الوزن واحد تقع على مستقيم واحد فقط من الوزن n بالأقواس الموزونة و النقاط من الوزن اثنين تقع على مستقيمين من الوزن n .