

Approximate solutions of H 's Variational iteration method for solving space-time fractional wave like and heat like equations

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Abstract:-

In this paper, we will consider H 's variational iteration method for solving space-time fractional wave like and heat like equations with appropriate data. This method is using linear operator with putting approximate value for time fractional derivative. Six examples are presented to show the application of the variational iteration method. This application shows important and efficient the method, which shows the approximate solutions near to exact solution.

Keywords: Variational iteration method, space-time fractional wave like and heat like equations, Caputo derivative.

1.Introduction

The variational iteration method has been extensively worked out for many years by numerous authors. Starting from the pioneer ideas of the Inokuti -Sekine-Mura method[1978], Ji-Huan He [1999] developed the variational iteration method . The variational iteration method, has been widely applied to solve nonlinear problems, more and more merits have been discovered and some modifications are suggested to overcome the demerits arising in the solution procedure. For example Jassim [2012] apply of variational iteration method by choosing linear operators for higher

dimensional initial value boundary problems, Schrödinger and Laplace problems. We introduce the basic idea underlying the variational iteration method for solving nonlinear equations. Consider the general equation:

$$L(x, y, z, t) + N(x, y, z, t) = g(x, y, z, t), (1)$$

where L is a linear differential operator, N is a nonlinear operator, and g is a given analytical function. The essence of the method is to construct a correction functional of the form

$$u_{n+1}(x, y, z, t) = u_n + \int_0^t \lambda(s, t) [Lu_n(x, y, z, s) + N\tilde{u}_n(x, y, z, s) - g(x, y, z, s)] ds, (2)$$

where λ is a Lagrange multiplier which can be identified optimally via the variational theory Inokuti and Sekine [1978], u_n is the approximate solution and $\tilde{u}_n$ denotes the restricted variation, i.e. $\delta\tilde{u}_n = 0$. After determining the Lagrange multiplier λ and selecting an appropriate initial function u_0 , the successive approximations u_n of the solution u can be readily obtained.

Definition 1.1. Fractional integral operator of order $\alpha \geq 0$ is defined as

$$I_x^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x - \tau)^{\alpha-1} f(\tau) d\tau \quad \alpha > 0, (3)$$

Γ is a gamma function.

Definition 1.2. Fractional derivative of $f(x)$ in the Caputo sense [Caputo,1967] is defined as

$$D_x^\alpha f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-\tau)^{m-\alpha-1} \frac{d^m f(\tau)}{d\tau^m} d\tau, \quad m-1 < \alpha \leq m, m \in \mathbb{N}, x > 0 \quad (4)$$

α is the order of the derivative. For the Caputo's derivative we have:

$$1 - D^\alpha C = 0, \quad C \text{ is constant,}$$

$$2 - D^\alpha x^\beta = 0, \quad \beta \leq \alpha - 1,$$

$$3 - D^\alpha x^\beta = \frac{\Gamma(1+\beta)}{\Gamma(1-\alpha+\beta)} x^{\beta-\alpha}, \quad \beta > \alpha - 1,$$

In this paper, we consider, the three dimensional space-time fractional wave like and heat like

$$\frac{\partial^\alpha u}{\partial t^\alpha} = f(x, y, z) \frac{\partial^\beta u}{\partial x^\beta} + g(x, y, z) \frac{\partial^\beta u}{\partial y^\beta} + h(x, y, z) \quad (5)$$

$$0 < x < a, \quad 0 < y < b, \quad 0 < z < c, \quad 0 < \alpha < 2, \quad 1 < \beta \leq 2, \quad t > 0.$$

Subject to the boundary conditions

$$u(0, y, z, t) = f_1(y, z, t), u_x(a, y, z, t) = f_2(y, z, t),$$

$$u(x, 0, z, t) = g_1(x, z, t), u_y(x, b, z, t) = g_2(x, z, t),$$

$$u(x, y, 0, t) = h_1(x, y, t), u_z(x, y, c, t) = h_2(x, y, t),$$

and the initial conditions

$$u(x, y, z, 0) = \psi(x, y, z), u_t(x, y, z, 0) = \eta(x, y, z),$$

where α and β are parameters describing the order of the fractional time- and space derivatives, respectively. In case $0 < \alpha \leq 1$, $1 < \beta \leq 2$, Equation (5) reduce to the fractional heat like equation with variable coefficients, and in case $1 < \alpha \leq 2$, $1 < \beta \leq 2$, Equation(5) reduce to the fractional wave like equation. In this paper, we apply H 's variational iteration method to find approximate solutions for fractional wave like and heat like equations.

2.H 's variational iteration method for solving space-time fractional wave like and heat like.

First, to convey the basic ideal for variational iteration method to solve fractional heat like equation, equation (5) can be written by using linear operator in the form:

$$\begin{aligned} & \frac{\partial^\alpha u(x, y, z, t)}{\partial t^\alpha} \\ &= k(x, y, z, t) - k(x, y, z, t) + [f(x, y, z) \frac{\partial^\beta u(x, y, z, t)}{\partial x^\beta} \\ &+ g(x, y, z) \frac{\partial^\beta u(x, y, z, t)}{\partial y^\beta} \\ &+ h(x, y, z) \frac{\partial^\beta u(x, y, z, t)}{\partial z^\beta}], 0 < \alpha \leq 1, 1 < \beta \leq 2, k \text{ is constant,} \end{aligned} \quad (6)$$

using the standard variational iteration method, we construct the following correction functional as

$$\begin{aligned} u_{n+1}(x, y, z, t) = & u_n(x, y, z, t) + \int_0^t \lambda(\xi, t) \left[\frac{\partial^\alpha u_n}{\partial \xi^\alpha} - k u_n(x, y, z, \xi) + k u_n(x, y, z, \xi) - \right. \\ & \left. f(x, y, z) \frac{\partial^\beta u_n}{\partial x^\beta} - g(x, y, z) \frac{\partial^\beta u_n}{\partial y^\beta} - h(x, y, z) \frac{\partial^\beta u_n}{\partial z^\beta} \right] d\xi, \end{aligned} \quad (7)$$

Now, we assume that

$$\frac{\partial^\alpha u}{\partial \xi^\alpha} \cong \Gamma(1 + \alpha) \frac{\partial}{\partial \xi}, \quad 0 < \alpha \leq 1, \quad (8)$$

if $\alpha = 1$, Equation (8) becomes $\frac{\partial}{\partial \xi} = \Gamma(2) \frac{\partial}{\partial \xi}$,

Substituting (8) in (7), we obtain

$$\begin{aligned} u_{n+1}(x, y, z, t) = & u_n(x, y, z, t) + \int_0^t \lambda(\xi, t) \left[\Gamma(1 + \alpha) \frac{\partial}{\partial \xi} - k u_n(x, y, z, \xi) \right. \\ & \left. + k u_n(x, y, z, \xi) - f(x, y, z) \frac{\partial^\beta u_n}{\partial x^\beta} - g(x, y, z) \frac{\partial^\beta u_n}{\partial y^\beta} - h(x, y, z) \frac{\partial^\beta u_n}{\partial z^\beta} \right] d\xi, \\ \delta u_{n+1}(x, y, z, t) = & \delta u_n(x, y, z, t) + \delta \int_0^t \lambda(\xi, t) \left[\Gamma(1 + \alpha) \frac{\partial u_n}{\partial \xi} - k u_n(x, y, z, \xi) + \right. \\ & \left. k \tilde{u}_n(x, y, z, s) - f(x, y, z) \frac{\partial^\beta \tilde{u}_n}{\partial x^\beta} - g(x, y, z) \frac{\partial^\beta \tilde{u}_n}{\partial y^\beta} - h(x, y, z) \frac{\partial^\beta \tilde{u}_n}{\partial z^\beta} \right] d\xi, \quad (9) \end{aligned}$$

$$\begin{aligned} \delta u_{n+1}(x, y, z, t) = & \delta u_n(x, y, z, t) + \Gamma(1 + \alpha) \lambda(\xi, t) \delta u_n(x, y, z, \xi) - \Gamma(1 + \alpha) \int_0^t \frac{\partial \lambda(\xi, t)}{\partial \xi} \\ & \delta u_n(x, y, z, \xi) d\xi - k \int_0^t \lambda(\xi, t) \delta u_n(x, y, z, \xi) d\xi. \quad (10) \end{aligned}$$

Moreover, the stationary conditions are as follow;

$$\Gamma(1 + \alpha) \left. \frac{\partial \lambda(\xi, t)}{\partial \xi} \right|_{\xi=t} + k \lambda(\xi, t) \Big|_{\xi=t} = 0,$$

$$1 + \Gamma(1 + \alpha)\lambda(\xi, t)|_{\xi=t} = 0,$$

therefore, the general Lagrange multiplier can be readily identified by

$$\lambda(\xi, t) = -\frac{1}{\Gamma(1+\alpha)} e^{-\frac{1}{\Gamma(1+\alpha)}K(\xi-t)}, \quad (11)$$

Substituting (11) for correction functional (7), we have the following iteration formula:

$$u_{n+1}(x, y, z, t) = u_n(x, y, z, t) - \frac{1}{\Gamma(1 + \alpha)} \int_0^t e^{-\frac{1}{\Gamma(1+\alpha)}K(\xi-t)} \left[\frac{\partial^\alpha u_n}{\partial \xi^\alpha} - f(x, y, z) \frac{\partial^\beta u_n}{\partial x^\beta} \right. \\ \left. - g(x, y, z) \frac{\partial^\beta u_n}{\partial y^\beta} - h(x, y, z) \frac{\partial^\beta u_n}{\partial z^\beta} \right] d\xi, \quad (12)$$

by the variational iteration formula (12) and initial approximation, we will get that first iterative step is the exact solution., when $\alpha = 1, \beta = 2$, as shows in this paper.

Second, to convey the basic ideal for variational iteration method to solve fractional wave like equation, Equation (5) can be written by using linear operation the form:

$$\frac{\partial^\alpha u(x, y, z, t)}{\partial t^\alpha} \\ = k(x, y, z, t) - k(x, y, z, t) + [f(x, y, z) \frac{\partial^\beta u(x, y, z, t)}{\partial x^\beta} \\ + g(x, y, z) \frac{\partial^\beta u(x, y, z, t)}{\partial y^\beta} \\ + h(x, y, z) \frac{\partial^\beta u(x, y, z, t)}{\partial z^\beta}], 1 < \alpha \leq 2, 1 < \beta \leq 2, k \text{ is constant}, (13)$$

using the standard variational iteration method, we construct the following correction functional as

$$u_{n+1}(x, y, z, t) = u_n(x, y, z, t) + \int_0^t \lambda(\xi, t) \left[\frac{\partial^\alpha u_n}{\partial \xi^2} - k u_n(x, y, z, \xi) + k u_n(x, y, z, \xi) \right]$$

$$f(x, y, z) \frac{\partial^\beta u_n}{\partial x^\beta} - g(x, y, z) \frac{\partial^\beta u_n}{\partial y^\beta} - h(x, y, z) \frac{\partial^\beta u_n}{\partial z^\beta} d\xi, (14)$$

Now, we assume that

$$\frac{\partial^\alpha u}{\partial \xi^\alpha} \square \Gamma(\alpha) \frac{\partial^2 u}{\partial \xi^2}, \quad 1 < \alpha \leq 2, \quad (15)$$

ii) $\alpha = 2$, Equation (15) becomes:

$$\frac{\partial^2 u}{\partial \xi^2} = \Gamma(2) \frac{\partial^2 u}{\partial \xi^2}.$$

Substituting (15) in (14), we obtain

$$u_{n+1}(x, y, z, t) = u_n(x, y, z, t) + \int_0^t \lambda(\xi, t) \left[\Gamma(\alpha) \frac{\partial^2 u}{\partial \xi^2} - k u_n(x, y, z, \xi) + k u_n(x, y, z, \xi) + \right.$$

$$\left. f(x, y, z) \frac{\partial^\beta u_n}{\partial x^\beta} + g(x, y, z) \frac{\partial^\beta u_n}{\partial y^\beta} + h(x, y, z) \frac{\partial^\beta u_n}{\partial z^\beta} \right] d\xi,$$

$$\delta u_{n+1}(x, y, z, t) = \delta u_n(x, y, z, t) + \delta \int_0^t \lambda(\xi, t) \left[\Gamma(\alpha) \frac{\partial^2 u}{\partial \xi^2} - k u_n(x, y, z, \xi) + \right.$$

$$\left. k \tilde{u}_n(x, y, z, \xi) - f(x, y, z) \frac{\partial^\beta u_n}{\partial x^\beta} - g(x, y, z) \frac{\partial^\beta u_n}{\partial y^\beta} - h(x, y, z) \frac{\partial^\beta u_n}{\partial z^\beta} \right] d\xi, (16)$$

$$\delta u_{n+1}(x, y, z, t)$$

$$= \delta u_n(x, y, z, t) - \Gamma(\alpha) \frac{\partial (\xi, t)}{\partial \xi} \delta u_n(x, y, z, \xi) + \Gamma(\alpha) \lambda(\xi, t) \frac{\partial u_n(x, y, z, \xi)}{\partial \xi} +$$

$$\Gamma(\alpha) \int_0^t \frac{\partial^2 \lambda(\xi, t)}{\partial \xi^2} \delta u_n(x, y, z, \xi) d\xi - k \int_0^t \lambda(\xi, t) \delta u_n(x, y, z, \xi) d\xi, (17)$$

moreover, the stationary conditions are as follow

$$\Gamma(\alpha) \frac{\partial^2 \lambda(\xi, t)}{\partial \xi^2} \Big|_{\xi=t} - k (\xi, t) \Big|_{\xi=t} = 0,$$

$$1 - \Gamma(\alpha) \frac{\partial (\xi, t)}{\partial \xi} \Big|_{\xi=t} = 0,$$

$$\Gamma(\alpha)\lambda(\xi, t)|_{\xi=t} = 0,$$

therefore, the general Lagrange multiplier can be readily identified by

$$\lambda(\xi, t) = \frac{1}{(\Gamma\alpha)^{1/2}} \sinh\left(\frac{k\xi - kt}{(\Gamma\alpha)^{1/2}}\right), (18)$$

Substituting (18) for correction functional (14), we have the following iteration formula:

$$u_{n+1}(x, y, z, t) = u_n(x, y, z, t) + \frac{1}{(\Gamma\alpha)^{1/2}} \int_0^t \sinh\left(\frac{k\xi - kt}{(\Gamma\alpha)^{1/2}}\right) \left[\frac{\partial^\alpha u_n}{\partial \xi^\alpha} - f(x, y, z) \frac{\partial^\beta u_n}{\partial x^\beta} - g(x, y, z) \frac{\partial^\beta u_n}{\partial y^\beta} - (x, y, z) \frac{\partial^\beta u_n}{\partial z^\beta} \right] d\xi, \quad (19)$$

by the variational iteration formula (19) and initial approximation, we will get that first iterative step is the exact solution, when $\alpha = 2, \beta = 2$, as shows in this paper.

3.Applications and results.

Example 3.1. We consider the one –dimensional space-time fractional heat -like equation Molliq, at el.[2009]:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{1}{2} x^2 \frac{\partial^\beta u}{\partial x^\beta}, \quad 0 < x < 1, 0 < \alpha \leq 1, 1 < \beta \leq 2, t > 0, \quad (20)$$

the initial condition

$$u(x, 0) = x^2, \quad (21)$$

The exact solution $\alpha = 1, \beta = 2$

$$u(x, t) = x^2 e^t, (22)$$

we can be written (20) in the form from Equation(6)

$$\frac{\partial^\alpha u(x, t)}{\partial t^\alpha} = \frac{1}{2} \frac{\partial^\beta u(x, t)}{\partial x^\beta} + u(x, t) - u(x, t), \text{ where } k = 1. (23)$$

We make the correction functional and the stationary conditions for Equation(23), the Lagrange multiplier can be determined as :

$$\lambda(\xi, t) = -\frac{1}{\Gamma(1+\alpha)} e^{-\frac{(\xi-t)}{\Gamma(1+\alpha)}} \quad (24)$$

$$u_{n+1}(x, t) = u_n(x, t) - \int_0^t \frac{1}{\Gamma(1+\alpha)} e^{-\frac{(\xi-t)}{\Gamma(1+\alpha)}} \left[\frac{\partial^\alpha u_n(x, \xi)}{\partial \xi^\alpha} - \frac{1}{2} x^2 \frac{\partial^\beta u_n(x, \xi)}{\partial x^\beta} \right] d\xi,$$

$$u_1(x, t) = u_0(x, t) - \int_0^t \frac{1}{\Gamma(1+\alpha)} e^{-\frac{(\xi-t)}{\Gamma(1+\alpha)}} \left[\frac{\partial^\alpha u_0(x, \xi)}{\partial \xi^\alpha} - \frac{1}{2} x^2 \frac{\partial^\beta u_0(x, \xi)}{\partial x^\beta} \right] d\xi,$$

$$u_1(x, t) = x^2 - \int_0^t \frac{1}{\Gamma(1+\alpha)} e^{-\frac{(\xi-t)}{\Gamma(1+\alpha)}} \left[\frac{\partial^\alpha x^2}{\partial \xi^\alpha} - \frac{1}{2} x^2 \frac{\partial^\beta x^2}{\partial x^\beta} \right] d\xi,$$

$$u_1(x, t) = x^2 + \left[-\frac{1}{2} x^2 \frac{\Gamma(3)}{\Gamma(3-\beta)} x^{2-\beta} + \frac{1}{2} x^2 \frac{\Gamma(3)}{\Gamma(3-\beta)} x^{2-\beta} e^{-\frac{t}{\Gamma(1+\alpha)}} \right],$$

$$u_1(x, t) = x^2 + \left[\frac{-x^\beta}{\Gamma(3-\beta)} + \frac{x^{4-\beta}}{\Gamma(3-\beta)} e^{-\frac{t}{\Gamma(1+\alpha)}} \right],$$

when $\alpha = 1, \beta = 2, u_1(x, t) = x^2 e^t$, is the exact solution.

Table.1. Approximate solutionsforexample(1) of $u_1(x, t)$.

t	x	$\alpha = 0.5, \beta = 2$	$\alpha = 0.5, \beta = 1.7$	$\alpha = 1, \beta = 1.5$	$\alpha = 1, \beta = 2$
0.2	0.	0.1193313959	0.1127744990	0.1057984836	0.11556228
5	3	0.4773255836	0.4721547886	0.4493697188	75
0.2	0.	1.0739825630	1.0949887500	1.0562739860	0.46224915
5	6	0.1582220228	0.1429713072	0.1260841380	01
0.2	0.	0.6328880910	0.6208613162	0.5641227091	1.04006058
5	9	1.4239982050	1.4728565880	1.3724960430	80
0.5	0.	0.2097872760	0.1830093884	0.1521314338	0.14838491
0.5	3	0.8391491039	0.8180319553	0.7114684653	44
0.5	0.	1.8880854840	1.9738732160	1.7785332010	0.59353965
0.7	6	0.2781578720	0.2360960560	0.1855768236	76
5	0.	1.1126314880	1.0794613730	0.9006641610	1.33546423
0.7	9	2.5034208480	2.6381733670	2.2998952310	00
5	0.				0.19053000
0.7	3				15
5	0.				0.76212000
1.0	6				61
1.0	0.				1.71477001
1.0	9				40
	0.				0.24464536
	3				45
	0.				0.97858145
	6				81
	0.				2.20180828
	9				10

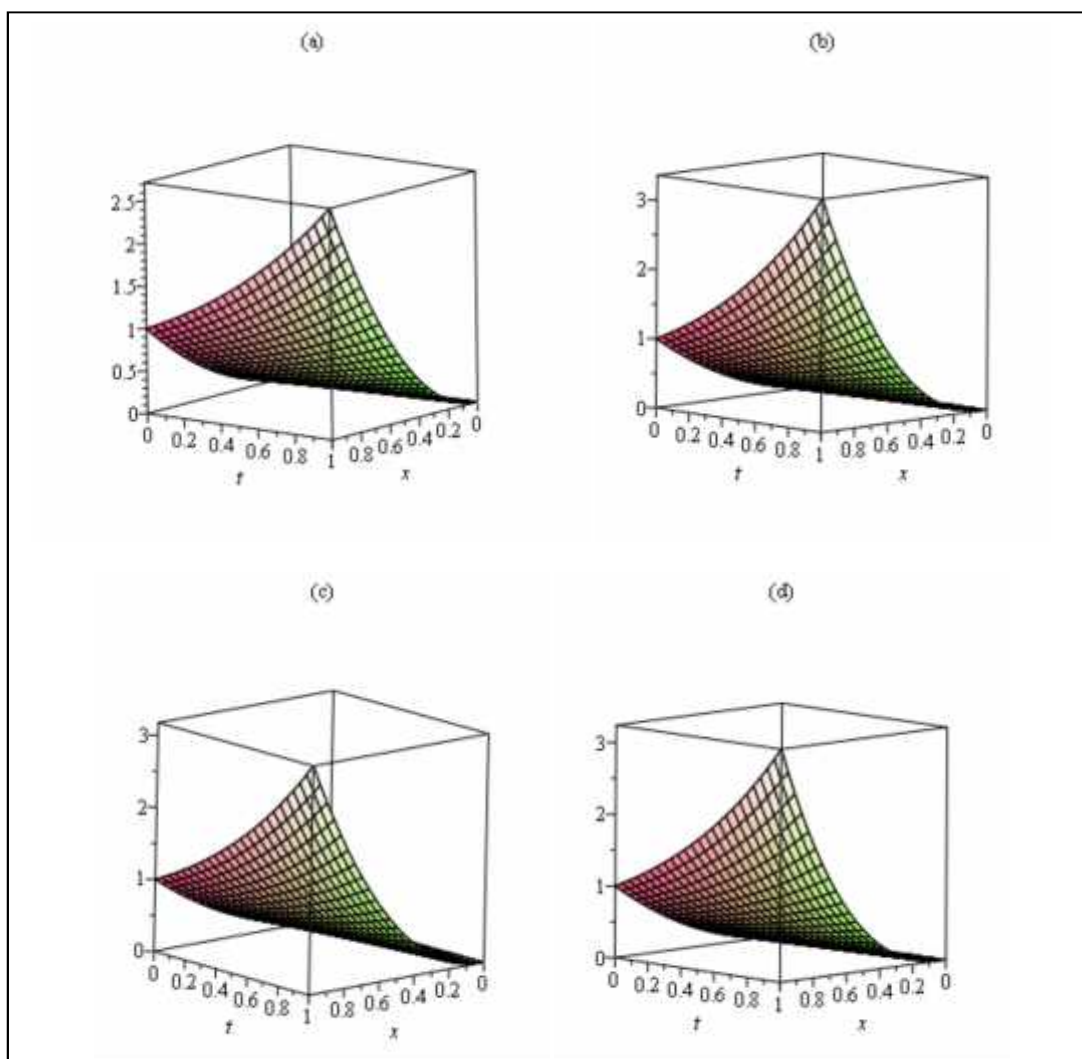


Fig.1 . The surface show the solution (x, t) for example (1)

$$u(x, t) = 1, \quad u(x, t) = 2, \quad u(x, t) = 0.5, \quad u(x, t) = 1.5$$

$$u(x, t) = 0.75, \quad u(x, t) = 1.75, \quad u(x, t) = 0.4, \quad u(x, t) = 1.2$$

Example.2. We consider the two-dimensional space-time fractional heat-like equation. Batiha, et al.[2009]:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^\beta u}{\partial x^\beta} + \frac{\partial^\beta u}{\partial y^\beta}, \quad 0 < x, y < 2\pi, \quad 0 < \alpha \leq 1, 1 < \beta \leq 2, t > 0, \quad (25)$$

with the initial condition

$$u(x, y, 0) = \sin x \sin y. \quad (26)$$

The exact solution when $(\alpha = 1, \beta = 2)$,

$$u(x, y, t) = \sin x \sin y e^{-2t},$$

we can be written (25) in the form:

$$\frac{\partial^\alpha u(x, y, t)}{\partial t^\alpha} = \frac{\partial^\beta u(x, y, t)}{\partial x^\beta} + \frac{\partial^\beta u(x, y, t)}{\partial y^\beta} - 2u(x, y, t) + 2u(x, y, t), \text{ where } k = -2, \quad (27)$$

we make the correction functional and the stationary conditions for Equation(27), the Lagrange multiplier can be determined as :

$$\lambda(\xi, t) = -\frac{1}{\Gamma(1+\alpha)} e^{\frac{2(\xi-t)}{\Gamma(1+\alpha)}}, \quad (28)$$

$$\begin{aligned} u_{n+1}(x, y, t) &= u_n(x, y, t) \\ &- \int_0^t \frac{1}{\Gamma(1+\alpha)} e^{\frac{2(\xi-t)}{\Gamma(1+\alpha)}} \left[\frac{\partial^\alpha u_n(x, y, \xi)}{\partial \xi^\alpha} - \frac{\partial^\beta u_n(x, y, \xi)}{\partial x^\beta} - \frac{\partial^\beta u_n(x, y, \xi)}{\partial y^\beta} \right] d\xi, \end{aligned}$$

$$\begin{aligned} u_1(x, y, t) &= u_0(x, y, t) \\ &- \int_0^t \frac{1}{\Gamma(1+\alpha)} e^{\frac{2(\xi-t)}{\Gamma(1+\alpha)}} \left[\frac{\partial^\alpha u_0(x, y, \xi)}{\partial \xi^\alpha} - \frac{\partial^\beta u_0(x, y, \xi)}{\partial x^\beta} - \frac{\partial^\beta u_0(x, y, \xi)}{\partial y^\beta} \right] d\xi, \end{aligned}$$

$$u_1(x, y, t) = \sin x \sin y - \int_0^t \frac{1}{\Gamma(1+\alpha)} e^{\frac{2(\xi-t)}{\Gamma(1+\alpha)}} \left[-\sin y \sin \left(x + \frac{\pi\beta}{2} \right) - \sin x \sin \left(y + \frac{\pi\beta}{2} \right) \right] d\xi,$$

$$u_1(x, y, t) = \sin x \sin y + \frac{1}{2} \left[\sin y \sin \left(x + \frac{\pi\beta}{2} \right) + \sin x \sin \left(y + \frac{\pi\beta}{2} \right) \right] (1 - e^{\frac{-2t}{\Gamma(1+\alpha)}}),$$

when $\alpha = 1, \beta = 2, u_1(x, y, t) = \sin x \sin y e^{-2t}$, is the exact solution.

Table.2. Approximate solutions for example (2) of $u_1(x, y, t)$

t	x	y	$\alpha = 0.5, \beta = 1.75$	$\alpha = 0.7, \beta = 1.5$	$\alpha = 1, \beta = 2$
0.1	$\pi/4$	$2\pi/3$	0.5080808599	0.544901347	0.50136814
0.1	$\pi/4$	$4\pi/3$	-	8	56
0.2	$\pi/4$	$2\pi/3$	0.5354144104	-	-
0.2	$\pi/4$	$4\pi/3$	0.4248586083	0.594293612	0.50136814
0.3	$\pi/4$	$2\pi/3$	-	3	56
0.3	$\pi/4$	$4\pi/3$	0.4740036962	0.490760459	0.41048551
0.4	$\pi/4$	$4\pi/3$	0.3584491875	0	93
0.4	$\pi/4$	$2\pi/3$	-	-	-
0.5	$\pi/4$	$4\pi/3$	0.4249993749	0.579786604	0.41048551
0.5	$\pi/4$	$2\pi/3$	0.3054560146	8	93
	$\pi/4$	$4\pi/3$	-	0.447316134	0.33607711
			0.3858950665	7	83
			0.2631686956	-	0.33607711
			-	0.568145733	-83
			0.3546907382	2	0.27515667
				0.412455064	21
				6	-
				-	0.27515667
				0.558804737	21
				7	0.22527922
				0.384481463	95
				3	-
				-	0.22527922
				0.551309233	95
				8	

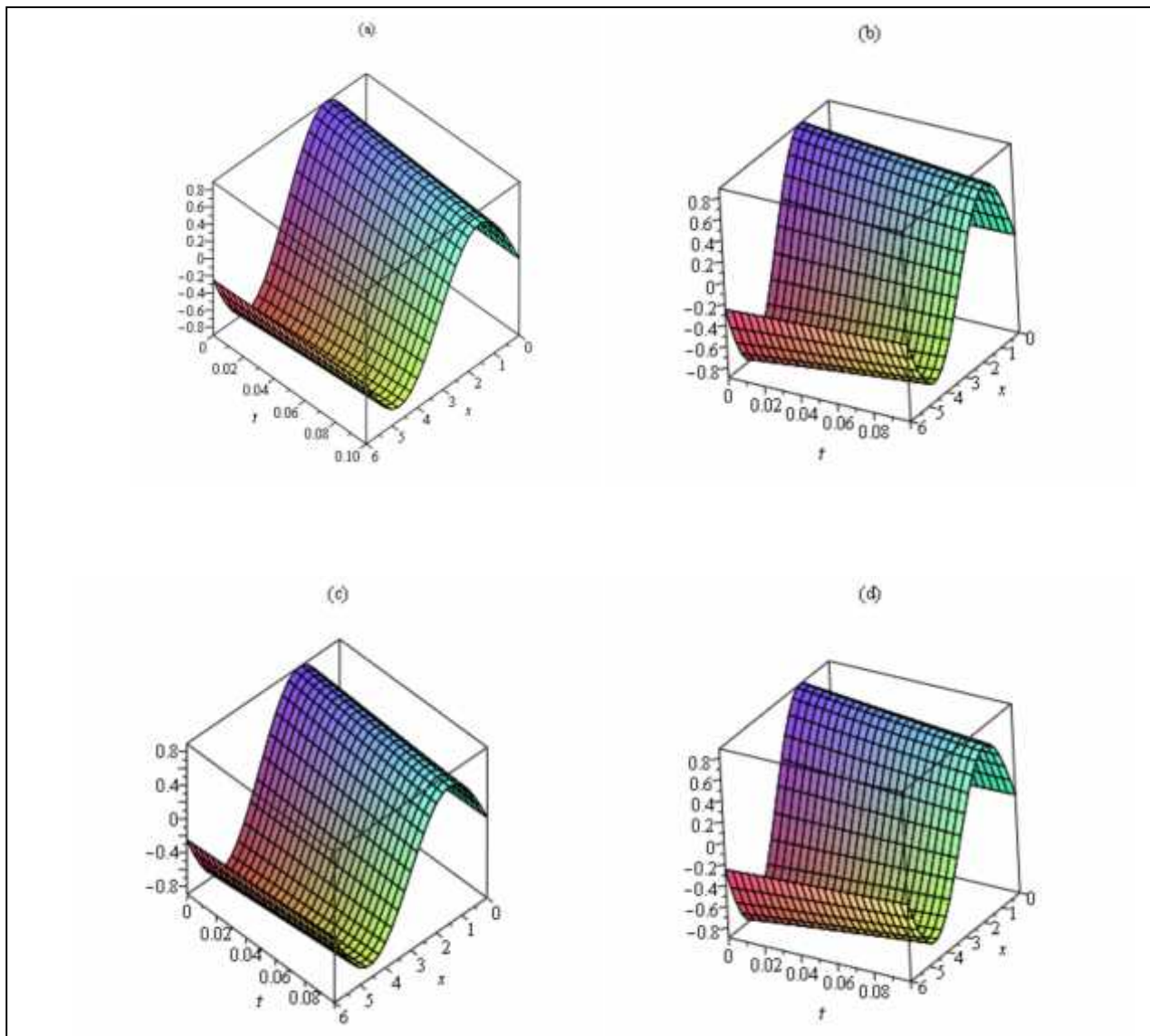


Fig.2 . The surface show the solution (x, t, u) for example (2)

$$u(0, t) = 1, \quad u(6, t) = 0.75, \quad u(6, 0) = 1.75$$

$$u(x, 0) = 0.5, \quad u(6, 0) = 1.5, \quad u(6, 0) = 0.5, \quad u(6, 0) = 1.75$$

Example.3. We consider the three–dimensional time fractional heat -like equation Molliq, at el[2009]:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = x^4 y^4 z^4 + \frac{1}{3} \left[x^2 \frac{\partial^2 u}{\partial x^2} + y^2 \frac{\partial^2 u}{\partial y^2} + z^2 \frac{\partial^2 u}{\partial z^2} \right], \quad 0 < x, y, z < 1, \quad 0 < \alpha \leq 1, \quad t > 0, \quad (29)$$

with the initial condition

$$u(x, y, z, 0) = 0. \quad (30)$$

The exact solution when $(\alpha = 1)$,

$$u(x, y, z, t) = x^4 y^4 z^4 (e^t - 1),$$

we can be written (29) in the form:

$$\begin{aligned} & \frac{1}{\Gamma(1+\alpha)} \int_0^t (t-\tau)^{\alpha-1} u(x, y, z, \tau) d\tau \\ &= x^4 y^4 z^4 \\ &+ \frac{1}{36} \left[\frac{1}{2} \frac{\partial^2}{\partial x^2} (x^4 y^4 z^4) + \frac{1}{2} \frac{\partial^2}{\partial y^2} (x^4 y^4 z^4) + \frac{1}{2} \frac{\partial^2}{\partial z^2} (x^4 y^4 z^4) \right] \\ &+ (x^4 y^4 z^4) - (x^4 y^4 z^4), \quad = 1. \quad (31) \end{aligned}$$

We make the correction functional and the stationary conditions for Equation(31), the Lagrange multiplier can be determined as :

$$(\lambda, \mu) = -\frac{1}{\Gamma(1+\alpha)} e^{-\frac{(t-\tau)}{\Gamma(1+\alpha)}}, \quad (32)$$

$$\begin{aligned} & \lambda_1(x, y, z, t) \\ &= u_n(x, y, z, t) \\ &- \int_0^t \frac{1}{\Gamma(1+\alpha)} e^{-\frac{(t-\tau)}{\Gamma(1+\alpha)}} \left[\frac{1}{2} \frac{\partial^2}{\partial x^2} (x^4 y^4 z^4) - \frac{1}{36} \left(\frac{1}{2} \frac{\partial^2}{\partial x^2} (x^4 y^4 z^4) \right. \right. \\ &\quad \left. \left. + \frac{1}{2} \frac{\partial^2}{\partial y^2} (x^4 y^4 z^4) + \frac{1}{2} \frac{\partial^2}{\partial z^2} (x^4 y^4 z^4) \right) \right] d\tau, \end{aligned}$$

$$\lambda_1(x, y, z, t) = \lambda_0(x, y, z, t) + \left[x^4 y^4 z^4 e^{\frac{t}{\Gamma(1+\alpha)}} - x^4 y^4 z^4 \right],$$

$$\lambda_1(x, y, z, t) = x^4 y^4 z^4 \left[e^{\frac{t}{\Gamma(1+\alpha)}} - 1 \right],$$

when $\alpha = 1$,

Table .3.Approximate solutionsforexample(3) of $_1(\ , \ , \ , \)$

				$= 0.7$	$= 0.9$	$= 1$
0.2	0.25	0.1	0.3	7.79033826610^{-10}	7.31381362010^{-10}	7.00532164010^{-10}
0.2	0.215	0.1	0.6	1.24645412210^{-8}	1.17021017910^{-8}	1.12085146210^{-8}
0.2	0.25`	0.5	0.3	4.86896141510^{-7}	4.57113351210^{-7}	4.37832602410^{-7}
0.2	0.25	0.5	0.6	0.000007790338266	0.000007313813620	0.000007005321640
0.5	0.25	0.1	0.3	2.32155960210^{-9}	2.15732389710^{-9}	2.05259464710^{-9}
0.5	0.25	0.1	0.6	3.71449536210^{-8}	3.45171823510^{-8}	3.28415143410^{-8}
0.5	0.25	0.5	0.3	0.000001450974751	0.000001348327435	0.000001282871654
0.5	0.25	0.5	0.6	0.000023215596020	0.000021573238970	0.000020525946470
0.8	0.25	0.1	0.3	4.46751617410^{-9}	4.10523829810^{-9}	3.87768809210^{-9}
0.8	0.25	0.1	0.6	7.14802587810^{-8}	6.56838127710^{-8}	6.20430094810^{-8}
0.8	0.25	0.5	0.3	0.000002792197608	0.000002565773936	0.000002423555057
0.8	0.25	0.5	0.6	0.000044675161740	0.000041052382980	0.000038776880920

$_1(\ , \ , \ , \) = {}^4 {}^4 {}^4 (\ - 1)$, is the exact solution.

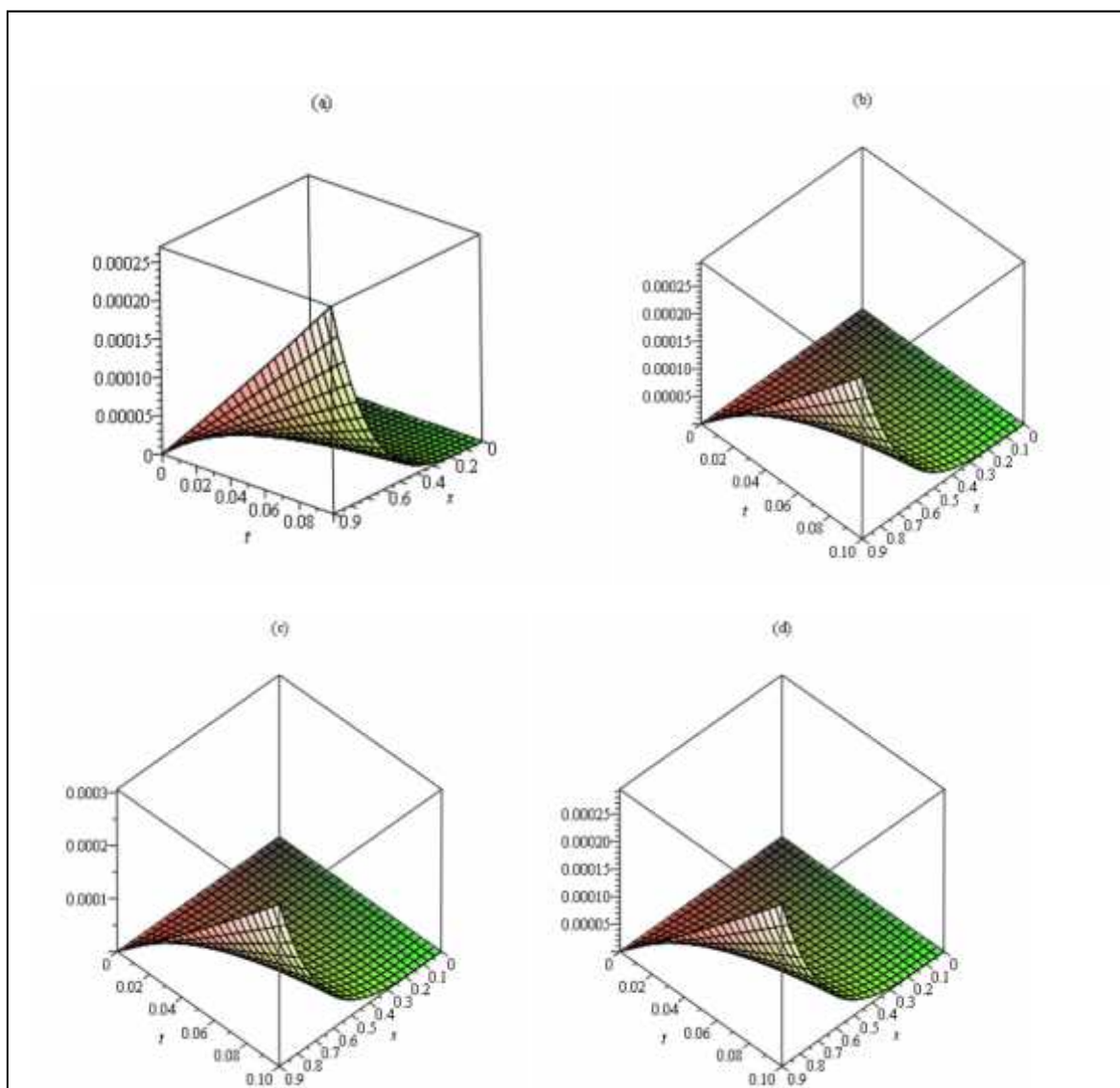


Fig.3 . The surface show the solution (u, v, w) for example(3), $y=z=0.5$

$$u(0) = 1, \quad v(0) = 0.75, \quad w(0) = 1.75$$

$$u(1) = 0.5, \quad v(1) = 1.5, \quad w(1) = 0.5, \quad w(1) = 1.75$$

Example .4. Consider the one-dimensional fractional wave-like equation Noor and

Mohyud[2008]:

$$u_t = \frac{1}{2} u^2, \quad 0 < \alpha < 1, 1 < \alpha \leq 2, \quad u > 0, (33)$$

with the initial condition

$$(\xi, 0) = \xi, \quad \frac{(\xi, 0)}{\Gamma(\alpha)} = \xi^2, \quad 0 < \alpha < 1, \quad (34)$$

the exact solution when $(\alpha = 2, \beta = 2)$

$$u(x, t) = x + x^2 \sinh t,$$

we can be written (33) in the form

$$\frac{\partial^\alpha u(x, t)}{\partial t^\alpha} = \frac{1}{2} x^2 \frac{\partial^\beta u(x, t)}{\partial x^\beta} + u(x, t) - u(x, t), \text{ where } k = 1. \quad (35)$$

We make the correction functional and the stationary conditions for Equation(35), the Lagrange multiplier can be determined as

$$\lambda(\xi, t) = \frac{1}{(\Gamma)^{1/2}} \sinh\left(\frac{\xi-t}{(\Gamma)^{\frac{1}{2}}}\right), \quad (36)$$

$$u_1(x, t) = u_0(x, t) + \int_0^t \frac{1}{(\Gamma\alpha)^{1/2}} \sinh\left(\frac{\xi-t}{(\Gamma\alpha)^{\frac{1}{2}}}\right) \left[\frac{\partial^\alpha u_0(x, \xi)}{\partial \xi^\alpha} - \frac{1}{2} x^2 \frac{\partial^\beta u_0(x, \xi)}{\partial x^\beta} \right] d\xi,$$

$$u_1(x, t) = x + x^2 t + \int_0^t \frac{1}{(\Gamma\alpha)^{1/2}} \sinh\left(\frac{\xi-t}{(\Gamma\alpha)^{\frac{1}{2}}}\right) \left[\frac{\partial^\alpha (x + x^2 \xi)}{\partial \xi^\alpha} - \frac{1}{2} x^2 \frac{\partial^\beta (x + x^2 \xi)}{\partial x^\beta} \right] d\xi,$$

$$u_1(x, t) = x + x^2 t - \frac{x^{4-\beta}}{\Gamma(3-\beta)} t + \frac{x^{4-\beta} (\Gamma\alpha)^{1/2}}{\Gamma(3-\beta)} \sinh\left(\frac{t}{(\Gamma\alpha)^{1/2}}\right).$$

when $\alpha = 2, \beta = 2, u_1(x, t) = x + x^2 \sinh t$, is the exact solution.

Table.4. Approximate solutions f_t (4) of $u_1(x, t)$

t	x	$\alpha = 1.5, \beta = 2$	$\alpha = 1.6, \beta = 1.75$	$\alpha = 1.5, \beta = 2$	$\alpha = 2, \beta = 2$
0.2	0.	0.322765398	0.3351007893	0.32276539	0.32273510
5	3	0	0.6988538720	80	85
0.2	0.	0.691061591	1.0597371790	0.69106159	0.69094043
5	6	9	0.3549947719	19	40
0.2	0.	1.104888582	0.7934861345	1.10488858	1.10461597
5	9	0	1.2953747600	20	70
0.5	0.	0.347145753	0.3775961165	0.34714575	0.34689857
0.5	3	7	0.9009968539	37	75
0.5	0.	0.788583014	1.5630801410	0.78858301	0.78759431
0.7	6	9	0.4044949888	49	00
5	0.	1.324311783	1.0289501750	1.32431178	1.32208719
0.7	9	0	1.8816883020	30	80
5	0.	0.374870588		0.37487058	0.37400850
0.7	3	6		86	59
5	0.	0.899482354		0.89948235	0.89603402
1.0	6	65		46	35
1.0	0.	1.573835298		1.57383529	1.56607655
1.0	9	0		80	30
	0.	0.407906678		0.40790667	0.40576810
	3	7		87	75
	0.	1.031626715		1.03162671	1.02307243
	6	0		50	00
	0.	1.871160108		1.87116010	1.85191296
	9	0		80	70

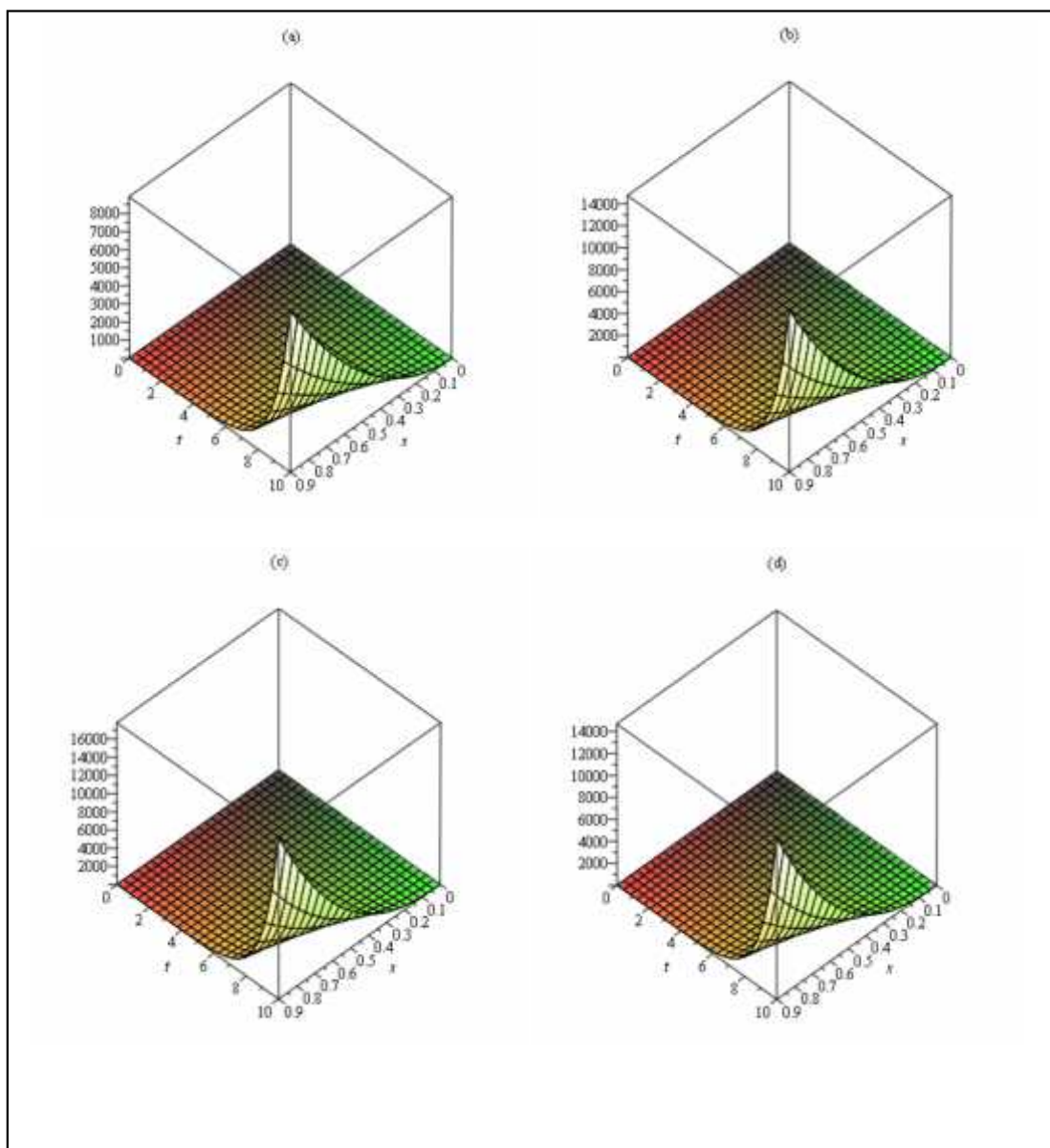


Fig.4 .The surface show the solution (u, v, w) for example (4).

$$(u, v, w) = 2, \quad (u, v, w) = 1.75, \quad (u, v, w) = 1.75$$

$$(u, v, w) = 1.5, \quad (u, v, w) = 1.5, \quad (u, v, w) = 1.5, \quad (u, v, w) = 1.75,$$

Example.5. We consider the three–dimensional space-time fractional wave -like equation (Molliq, *et al.* 2009)©

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{1}{1} \left[x^2 \frac{\partial^\beta u}{\partial x^\beta} + y^2 \frac{\partial^\beta u}{\partial y^\beta} \right], 0 < x, y, z < 1, 1 < \alpha, \beta \leq 2, t > 0, (37)$$

with the initial condition

$$u(x, y, 0) = x^4, u_t(x, y, 0) = y^4, (38)$$

the exact solution when $(\alpha = 2, \beta = 2)$,

$$u(x, y, t) = x^4 \cosh t + y^4 \sinh t,$$

we can be written (37) in the form

$$\frac{\partial^\alpha u(x, y, t)}{\partial t^\alpha} = \frac{1}{1} x^2 \frac{\partial^\beta u(x, y, t)}{\partial x^\beta} + \frac{1}{1} y^2 \frac{\partial^\beta u(x, y, t)}{\partial y^\beta} + u(x, y, t) - u(x, y, t), \text{ where } k = 1. (39)$$

We make the correction functional and the stationary conditions for Equation(39), the Lagrange multiplier can be determined as :

$$\lambda(\xi, t) = \frac{1}{(\Gamma)^{1/2}} \sinh\left(\frac{\xi-t}{(\Gamma)^{1/2}}\right), (40) u_1(x, y, t) = u_0 + \int_0^t \frac{1}{(\Gamma\alpha)^{1/2}} \sinh\left(\frac{\xi-t}{(\Gamma\alpha)^{1/2}}\right) \left[\frac{\partial^\alpha u_0(x, y, \xi)}{\partial \xi^\alpha} - \frac{1}{1} \left(x^2 \frac{\partial^\beta u_0(x, y, \xi)}{\partial x^\beta} + y^2 \frac{\partial^\beta u_0(x, y, \xi)}{\partial y^\beta} \right) \right] d\xi,$$

$$u_1 = x^4 + y^4 t$$

$$+ \int_0^t \frac{1}{(\Gamma\alpha)^{1/2}} \sinh\left(\frac{\xi-t}{(\Gamma\alpha)^{1/2}}\right) \left[\frac{\partial^\alpha (x^4 + y^4 \xi)}{\partial \xi^\alpha} - \frac{1}{12} \left(x^2 \frac{\partial^\beta (x^4 + y^4 \xi)}{\partial x^\beta} + y^2 \frac{\partial^\beta (x^4 + y^4 \xi)}{\partial y^\beta} \right) \right] d\xi,$$

$$u_1 = x^4 + y^4 t + \int_0^t \frac{1}{(\Gamma\alpha)^{1/2}} \sinh\left(\frac{\xi-t}{(\Gamma\alpha)^{1/2}}\right) \left[0 - \frac{1}{1} \left(\frac{4}{\Gamma(5-\beta)} x^{6-\beta} + \xi \frac{4}{\Gamma(5-\beta)} y^{6-\beta} \right) \right] d\xi,$$

$$u_1 = x^4 + y^4 t$$

$$- \left[\frac{\Gamma 3 x^{6-\beta}}{\Gamma(5-\beta)} \right]$$

$$- \frac{\Gamma 3 x^{6-\beta}}{\Gamma(5-\beta)} \cosh\left(\frac{t}{(\Gamma\alpha)^{1/2}}\right) + \frac{\Gamma(3)}{\Gamma(5-\beta)} y^{6-\beta} t - \frac{\Gamma(3)(\Gamma\alpha)^{1/2}}{\Gamma(5-\beta)} y^{6-\beta} \sinh\left(\frac{t}{(\Gamma\alpha)^{1/2}}\right) \Bigg],$$

Table.5. Approximate solutions f_t (5) of u_1

when $\alpha = 2, \beta = 2, u_1(x, y, t) = x^4 \cos ht + y^4 \sinh t$, is the exact solution

t	x	y	$\alpha = 1.5, \beta = 2$	$\alpha = 1.75, \beta = 1.6$	$\alpha = 1.7, \beta = 1.9$	$\alpha = 2, \beta = 2$
0.	0.0	0.2	0.000787150478		0.000785811419	0.00078647896
2	1	5	7		2	1
0.	0.0	0.7	0.063758370650	0.0007834483526	0.063692724170	0.06370397974
2	1	5	0	0.0635563319100	0	0
0.	0.0	0.2	0.000793531830		0.000792144421	0.00079284417
2	5	5	9	0.0007897159584	6	7
0.	0.0	0.7	0.063764752000	0.0635625995100	0.063699057170	0.06371034496
2	5	5	0	0.0019877205200	0	0
0.	0.0	0.2	0.002046268116	0.1625504995000	0.002025075975	0.00203553981
5	1	5	0	0.0019941363700	0	4
0.	0.0	0.7	0.165746801800	0.1625569154000	0.164706957100	0.16487782280
5	1	5	0	0.0032697062880	0	0
0.	0.0	0.2	0.002053409137	0.2713132109000	0.002031908513	0.00204257620
5	5	5	0	0.0032764125210	0	0
0.	0.0	0.7	0.165753942900	0.2713199172000	0.164713789500	0.16488485920
5	5	5	0		0	0
0.	0.0	0.2	0.003514957323		0.164713789500	0.00346917736
8	1	5	0		0	6
0.	0.0	0.7	0.284710436400		0.280341854600	0.28100229680
8	1	5	0		0	0
0.	0.0	0.2	0.003523589372		0.003433875183	0.00347752296
8	5	5	0		0	0
0.	0.0	0.7	0.284719068500		0.280349666200	0.28101064240
8	5	5	0		0	0

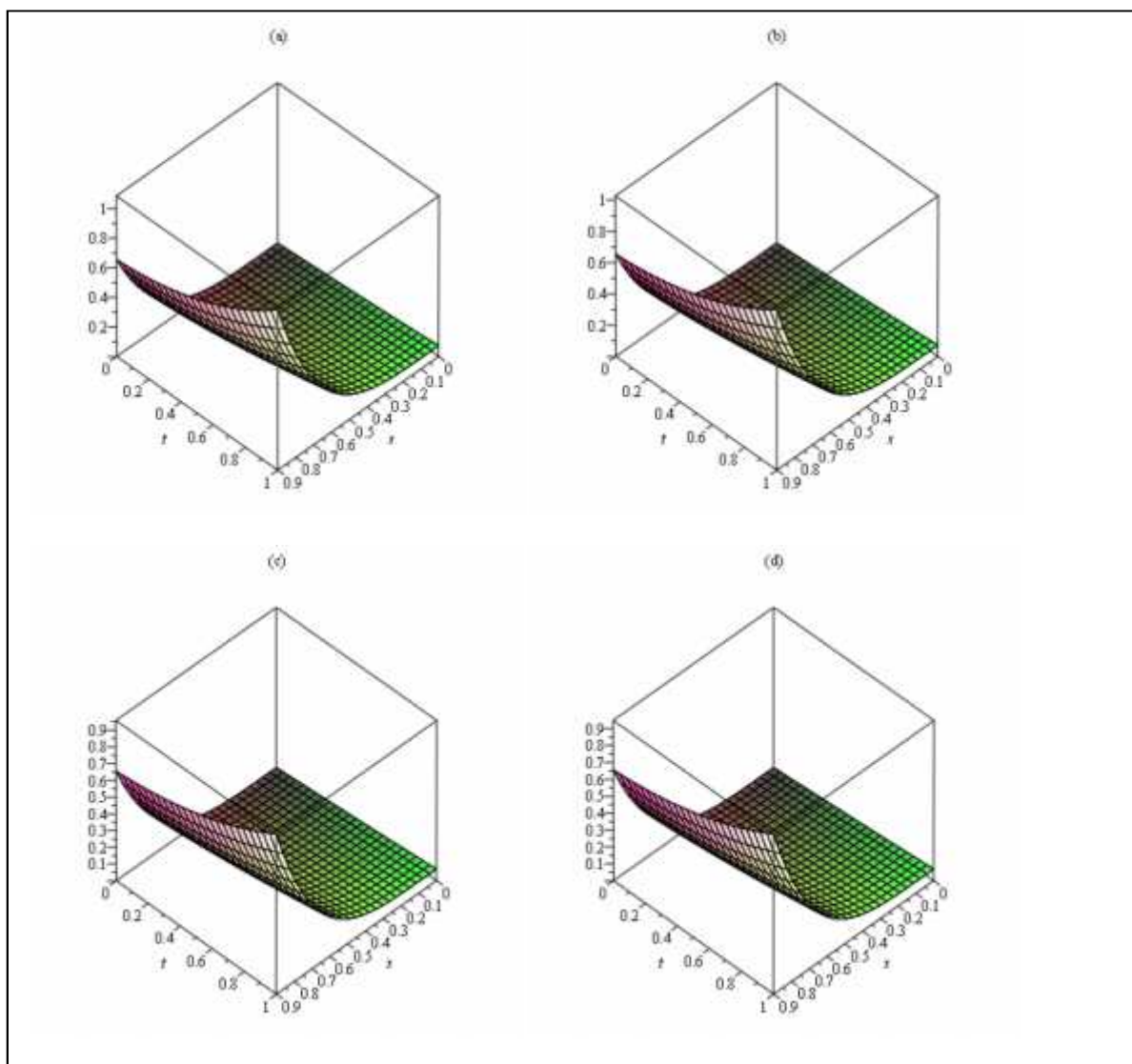


Fig.5 .The surface show the solution (u, v, w) for example (5),
 $u(0, t) = 0.5$, $v(0, t) = 2$, $w(0, t) = 2$, $u(1, t) = 1.75$, $v(1, t) = 1.75$
 $u(2, t) = 1.5$, $v(2, t) = 1.5$, $w(2, t) = 1.75$, $u(3, t) = 1.5$,

Example.6. We consider the three–dimensional space-time fractional wave -like Equation

$$(\text{Molliq, et al. 2009}) : \frac{\partial^\alpha u}{\partial t^\alpha} = x^2 + y^2 + z^2 + \frac{1}{2} \left[x^2 \frac{\partial^\beta u}{\partial x^\beta} + y^2 \frac{\partial^\beta u}{\partial y^\beta} + z^2 \frac{\partial^\beta u}{\partial z^\beta} \right],$$

$$0 < x, y, z < 1, 1 < \alpha, \beta \leq 2, t > 0, (41)$$

the initial condition

$$u(x, y, z, 0) = 0, u_t(x, y, z, 0) = x^2 + y^2 - z^2, \quad (42)$$

the exact solution when $(\alpha = 2, \beta = 2)$,

$$u(x, y, z, t) = -(x^2 + y^2 + z^2) + (x^2 + y^2)e^t + z^2e^{-t},$$

we can be written (41) in the form

$$\begin{aligned} & \frac{\partial^\alpha u(x, y, z, t)}{\partial t^\alpha} \\ &= x^2 + y^2 + z^2 + \frac{1}{2}x^2 \frac{\partial^\beta u(x, y, z, t)}{\partial x^\beta} + \frac{1}{2}y^2 \frac{\partial^\beta u(x, y, z, t)}{\partial y^\beta} + \frac{1}{2}z^2 \frac{\partial^\beta u(x, y, z, t)}{\partial z^\beta} \\ &+ \\ &u(x, y, z, t) - u(x, y, z, t), \quad (43) \end{aligned}$$

We make the correction functional and the stationary conditions for Equation(43), the Lagrange multiplier can be determined as :

$$\lambda(\xi, t) = \frac{1}{(\Gamma)^{1/2}} \sinh\left(\frac{\xi-t}{(\Gamma)^{\frac{1}{2}}}\right),$$

$$\begin{aligned} (44) u_{n+1}(x, y, z, t) = & u_n(x, y, z, t) + \int_0^t \frac{1}{(\Gamma\alpha)^{1/2}} \sinh\left(\frac{\xi-t}{(\Gamma\alpha)^{\frac{1}{2}}}\right) \left[\frac{\partial^\alpha u_n(x, y, z, \xi)}{\partial \xi^\alpha} - x^2 - y^2 - z^2 - \right. \\ & \left. \frac{1}{2}x^2 \frac{\partial^\beta u_n(x, y, z, \xi)}{\partial x^\beta} - \frac{1}{2}y^2 \frac{\partial^\beta u_n(x, y, z, \xi)}{\partial y^\beta} - \frac{1}{2}z^2 \frac{\partial^\beta u_n(x, y, z, \xi)}{\partial z^\beta} \right] d\xi, \end{aligned}$$

$$\begin{aligned} u_1(x, y, z, t) = & u_0(x, y, z, t) + \int_0^t \frac{1}{(\Gamma\alpha)^{1/2}} \sinh\left(\frac{\xi-t}{(\Gamma\alpha)^{\frac{1}{2}}}\right) \left[\frac{\partial^\alpha u_0(x, y, z, \xi)}{\partial \xi^\alpha} - x^2 - y^2 - z^2 - \right. \\ & \left. \frac{1}{2}x^2 \frac{\partial^\beta u_0(x, y, z, \xi)}{\partial x^\beta} - \frac{1}{2}y^2 \frac{\partial^\beta u_0(x, y, z, \xi)}{\partial y^\beta} - \frac{1}{2}z^2 \frac{\partial^\beta u_0(x, y, z, \xi)}{\partial z^\beta} \right] d\xi, \end{aligned}$$

$$u_1 = (x^2 + y^2 - z^2)t + \int_0^t \frac{1}{(\Gamma\alpha)^{\frac{1}{2}}} \sinh\left(\frac{\xi-t}{(\Gamma\alpha)^{\frac{1}{2}}}\right) \left[-\frac{x^{4-\beta}}{\Gamma(3-\beta)} \xi - x^2 - y^2 - z^2 - \right.$$

$$\left[\frac{y^{4-\beta}}{\Gamma(3-\beta)} \xi + \frac{z^{4-\beta}}{\Gamma(3-\beta)} \xi \right] d ,$$

$$u_1 = (x^2 + y^2 - z^2)t - (x^2 + y^2 + z^2) + (x^2 + y^2 + z^2) \cosh\left(\frac{t}{(\Gamma\alpha)^{\frac{1}{2}}}\right) - \frac{x^{4-\beta}}{\Gamma(3-\beta)}t - \frac{y^{4-\beta}}{\Gamma(3-\beta)}t + \frac{z^{4-\beta}}{\Gamma(3-\beta)}t + (x^{4-\beta} + y^{4-\beta} - z^{4-\beta}) \frac{(\Gamma\alpha)^{\frac{1}{2}}}{\Gamma(3-\beta)} \sinh\left(\frac{t}{(\Gamma\alpha)^{\frac{1}{2}}}\right),$$

when $\alpha = 2, \beta = 2, u_1(x, y, z, t) = -(x^2 + y^2 + z^2) + (x^2 + y^2 + z^2)e^{-t}$, is the exact solution.

Table.6. Approximate solutions for example (6) of $u_1(x, y, z, t)$.

t	x	y	z	$\alpha = 1.75, \beta = 1.6$	$\alpha = 1.7, \beta = 1.9$	$\alpha = 2, \beta = 2$
0.2	0.025	0.04	0.04	0.0002087734958	0.0002101692817	0.0002025903436
0.2	0.025	0.04	0.08	-0.003204049930	-0.000649825270	-0.000667502040
0.2	0.025	0.08	0.04	0.0038312642390	0.0012821025050	0.0012653235840
0.2	0.025	0.08	0.08	0.0004184408056	0.0004222506927	0.0003952312020
0.5	0.025	0.04	0.04	0.0008483322835	0.0008614353879	0.0008138538820
0.5	0.025	0.04	0.08	-0.004126013960	-0.000956096830	-0.001074798900
0.5	0.025	0.08	0.04	0.0071582236540	0.0040301836960	0.0039277159800
0.5	0.025	0.08	0.08	0.0021838774140	0.0022126514700	0.0020390631460
0.8	0.025	0.04	0.04	0.0019263764320	0.0019722092520	0.0018457549070
0.8	0.025	0.04	0.08	-0.004179587650	-0.000465537360	-0.000797466060
0.8	0.025	0.08	0.04	0.0115734036800	0.0079939994390	0.0077283513630
0.8	0.025	0.08	0.08	0.0054674395830	0.0055562528380	0.005081303900

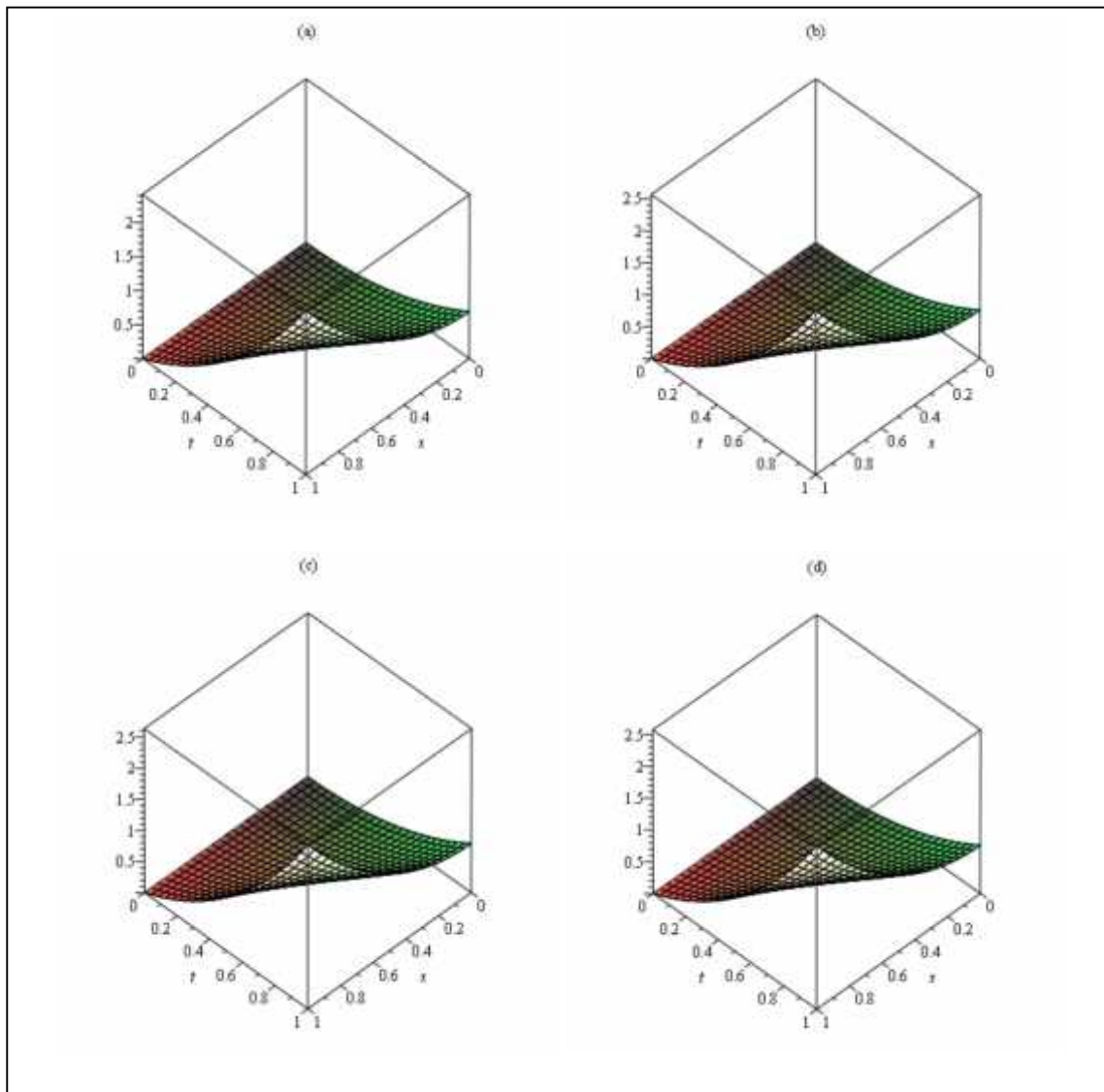


Fig.6 .The surface show the solution (x, t, z) for example (6), $\alpha = 0.8, z = 0.8$
 $u_1(x, t) = 2, u_2(x, t) = 2, u_3(x, t) = 1.75, u_4(x, t) = 1.75$
 $u_5(x, t) = 1.5, u_6(x, t) = 1.5, u_7(x, t) = 1.75, u_8(x, t) = 1.5$

4. conclusion

In this work, the variational iteration method is use to solve the fractional heat like and wave like. This technique provides the solution with first step i-e $u_1(x, t)$ is the exact solution in case $\alpha = \beta = 2$ in equation wave like and $\alpha = 1, \beta = 2$ in equation heat like. Results show the ability and efficiency of this method.

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طريقة أسلوب التغيرات التكراري لحل معادلاتي الحرارة والموجة الخاصة الكسرية للزمن والفضاء

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في هذا البحث سوف نطبق طريقة أسلوب التغيرات التكراري لحل معادلة الحرارة الخاصة والموجة الخاصة الكسرية الزمن

والفضاء. وفي هذه الطريقة تستخدم معامل خطي مع وضع قيمة تقريبية للمشتقة الكسرية الزمنية. ستة أمثلة تبرهن

تطبيق هذه الطريقة. هذه التطبيقات بينت أهمية وكفاءة الطريقة لان الحلول التقريبية ظهرت قريبة جدا من الحل

الحقيقي.

الكلمات المفتاحية: طريقة أسلوب التغيرات التكراري، معادلة الحرارة الخاصة الكسرية والموجة الخاصة الكسرية، مشتقة كباتو.