

On Weighted Arcs of Type $(n - q, n)$ in $PG(2, q)$

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Abstract

In this paper we proved that the existence of minimal $(4q - 2, q + 4; f)$ – arc of type $(4, q + 4)$, which having W is minimal and there is no point of weight greater than 2 in the projective plane of order q . Also we proved that the existence of $(q + 1 + \frac{q(q+1)}{2}, 2q + 1; f)$ – arc of type $(q + 1, 2q + 1)$ in $PG(2, q)$ where the points of weight 1 formed a conic, the points of weight 2 are the exterior points of the conic, the points of weight 0 are the interior points of the conic, W is minimal and $\text{Im}f = \{0, 1, 2\}$.

Key words. Projective Plane of order q , weighted (k, n) – arcs.

Mathematics Subject Classification. 51E21.

Introduction

[A. Barlotti, 1978] presented the notion of a $(k, n; \{w_i\})$ – set of kind s . The $(k, n; \{w_i\})$ – set of kind 2 in a projective plane, also called $(k, n; \{w_i\})$ – arcs, where studied by [M. Barnabei, 1979]. [E. D'Agostini, 1980], was developed the notion of $(k, n; f)$ – arcs of type $(n - 2, n)$ in $PG(2, q)$, where q is even. Also [B. J. Wilson, 1986], generalized the work of [E. D'Agostini, 1980], into $(k, n; f)$ – arcs of type $(n - 3, n)$ in $PG(2, q)$, where $3|q$. [F. K. Hameed, 1989], [F. K. Hameed et. al, 2011], [M. Y. Abass, 2011] and [R. D. Mahmood, 1990], continue on the same idea of [B. J. Wilson, 1986]. The notion of $(k, n; f)$ – arcs of type $(1, n)$ introduced by [G. Raguso and L. Rella, 1983].

1. Preliminaries

We will denote by $PG(2, q)$ the projective desarguesian plane of order $q = p^h$, by $\mathcal{P}$ the set of all points of the plane and by $\mathcal{R}$ the set of all lines of the plane. Then $PG(2, q)$ having $q^2 + q + 1$ points and $q^2 + q + 1$ lines. Each line contains $q + 1$ points and through every point there pass $q + 1$ lines. Also the projective plane has the following properties:

- (i) Any two points have exactly one line joining them and any two different lines meet at just one point.
- (ii) There exists four points, no three of them are collinear.

Definition 1.1 [J. W. P. Hirschfeld, 1998]. A (k, n) – arc $\mathcal{K}$ in $PG(2, q)$ is a set of k points no $n + 1$ of them are collinear, where $n \geq 2$. We will write simply k – arc for $(k, 2)$ – arc.

Definition 1.2 [J. W. P. Hirschfeld, 1998]. A line ℓ in $PG(2, q)$ is an i – secant of a (k, n) – arc $\mathcal{K}$ if $|\ell \cap \mathcal{K}| = i$. Let τ_i denote the total number of i – secants to $\mathcal{K}$ in $PG(2, q)$, then the type of $\mathcal{K}$ is defined by $(\tau_0, \tau_1, \dots, \tau_n)$.

Lemma 1.1 [J. W. P. Hirschfeld, 1998]. For a (k, n) – arc $\mathcal{K}$, the following equations hold:

- (i) $\sum_{i=0}^n \tau_i = q^2 + q + 1$;
- (ii) $\sum_{i=1}^n i\tau_i = k(q + 1)$;
- (iii) $\sum_{i=2}^n \frac{i(i-1)}{2} \tau_i = \frac{k(k-1)}{2}$.

From [J. W. P. Hirschfeld, 1998], we have 2 – secant of the conic $((q + 1) - \text{arc})$ having 2 points of the conic, $\frac{q-1}{2}$ exterior points (the points of intersection of two 1 – secants of the conic) and $\frac{q+1}{2}$ interior points (the points which there pass through them no 1 – secant of the conic). The 1 – secants of the conic contain q exterior points and 1 point of the conic. The 0 – secants of the conic contain $\frac{q+1}{2}$ exterior points and $\frac{q+1}{2}$ interior points.

For any function f from $\mathcal{P}$ to the set of natural numbers $\mathbb{N}$ we will say that $f(P)$ is the weight of the point P . By using f , we may define a function F from $\mathcal{R}$ to $\mathbb{N}$ in the following way:

$$F(r) = \sum_{P \in r} f(P)$$

and we will say that $F(r)$ is the weight of the line r . Moreover, if $F(r) = j$ we will also say that r is a j – weighting line.

Definition 1.3 [E. D'Agostini, 1994]. A $(k, n; f)$ – arc K in $PG(2, q)$ is a function

$f : \mathcal{P} \rightarrow \mathbb{N}$ such that $k = |\text{support of } f|$ (the points of non – zero weight) and $n = \max F$.

Let us remark that an ordinary (k, n) – arc is a $(k, n; f)$ – arc with $\text{Im } f = \{0, 1\}$.

Definition 1.4 [F. K. Hameed, 1989]. For any $(k, n; f)$ – arc K the underlying arc is the ordinary arc whose points are all the points of K .

Definition 1.5 [F. K. Hameed, 2012]. A $(k, n; f)$ - arc K is called monoidal if $\text{Im } f = \{0, 1, \omega\}$ and $l_\omega = 1$.

Definition 1.6 [E. D'Agostini, 1994]. The characters of a $(k, n; f)$ - arc K are the integers $t_j = |F^{-1}(j)|$ $j = 0, 1, \dots, n$.

Definition 1.7 [E. D'Agostini, 1994]. The type of a $(k, n; f)$ - arc K is the set of $\text{Im } F$. To write explicitly the type of K we can use the sequence $(n_1, \dots, n_\rho)$, where $n_\lambda \in \text{Im } F$, $\lambda = 1, \dots, \rho$ and $n_1 < n_2 < \dots < n_\rho = n$.

Let us use the following notations:

$\omega = \max f$, $W = \sum_{P \in \mathcal{P}} f(P)$ and W will be called the weight of K .

$L_i = f^{-1}(i)$ and $l_i = |L_i|$, $i = 0, 1, \dots, \omega$.

$[M]$ indicates the set of all lines through the point M .

It is well known from [E. D'Agostini, 1994], that:

$$k = \sum_{i=1}^{\omega} l_i \quad (1.1)$$

$$W = \sum_{i=1}^{\omega} i l_i \quad (1.2)$$

$$\sum_{r \in [M]} F(r) = W + q f(M) \quad (1.3)$$

$$|\text{Im } F| \geq 2 \quad (1.4)$$

A useful result, mentioned in [E. D'Agostini, 1994], is the following:

If there exists a point P of a $(k, n; f)$ - arc K such that every line through it is a n -weighting line, then

$$P \in L_\omega \quad (1.5)$$

If $M \in L_\omega$ and $u \in [M]$ then,

$$F(u) = n. \quad (1.6)$$

$$\text{Hence } W \leq (n - \omega)q + n.$$

An arc with weight such that the equality holds, is called maximal. Of course, a maximal arc is also such that through a point of maximal weight there pass only n -weighting lines.

Finally, we shall recall [E. D'Agostini, 1994] the following relations concerning the characters of $(k, n; f)$ - arc K :

$$\sum_{j=0}^n t_j = q^2 + q + 1 \quad (1.7)$$

$$\sum_{j=1}^n j t_j = (q + 1)W \quad (1.8)$$

$$\sum_{j=2}^n \binom{j}{2} t_j = \binom{W}{2} + q \sum_{i=2}^{\omega} \binom{i}{2} l_i \quad (1.9)$$

2. $(k, n; f)$ - arcs of type (m, n)

From now on, K shall denote a $(k, n; f)$ -arc of type (m, n) , where $|Im f| \geq 3$. Let firstly state the following:

Lemma 2.1 [F. K. Hameed, 1989]. The weight W of a $(k, n; f)$ -arc of type (m, n) satisfies:

$$m(q+1) \leq W \leq (n-\omega)q + n.$$

We call arcs for which the values in Lemma 2.1 are attained, maximal and minimal $(k, n; f)$ -arcs of type (m, n) respectively.

Theorem 2.1 [F. K. Hameed, 1989]. Let K be a $(k, n; f)$ -arc of type (m, n) , $m > 0$ and let v_m^s and v_n^s respectively the number of lines of weight m and the number of lines of weight n passing through a point of weight s . Then

$$\begin{aligned} (n-m)v_m^s &= (n-s)(q+1) - (W-s); \\ (n-m)v_n^s &= (W-s) - (m-s)(q+1). \end{aligned}$$

Theorem 2.2 [F. K. Hameed, 1989]. The necessary condition for the existence of a $(k, n; f)$ -arc K of type (m, n) , $m > 0$ is that:

- (i) $q \equiv 0 \pmod{n-m}$;
- (ii) $\omega \leq n-m$;
- (iii) $m \leq n-2$.

3. $(k, n; f)$ -arcs of type $(n-q, n)$

From [F. K. Hameed, 1989], we have the following:

- (i) $\sum_{r \in [M]} F(r) = W + q f(M)$;
- (ii) $(n-q)(q+1) \leq W \leq (n-\omega)q + n$.

Lemma 3.1. In the $(k, n; f)$ -arcs of type $(n-q, n)$ in $PG(2, q)$, if $W = (n-q)(q+1)$, then

- (i) $v_{n-q}^s = (q-s) + 1$;
 $v_n^s = s$.

For $s = 0, 1, 2, \dots, q$.

- (ii) $t_n = n-q$;
 $t_{n-q} = q^2 + 2q + 1 - n$.

Proof. (i) We have $v_{n-q}^s + v_n^s = q+1$ and if $f(M) = s$, then

$$(n-q)v_{n-q}^s + nv_n^s = \sum_{r \in [M]} F(r) = W + qs.$$

Solve these two equations when $W = (n-q)(q+1)$, give the result.

(ii) From the following equations

$$\sum_{j=0}^n t_j = q^2 + q + 1;$$

$$\sum_{j=1}^n j t_j = (q+1)W$$

For $j = n - q$, n we obtain the result. $\square$

Now, we take the case in which $l_0 > 0$, $l_1 > 0$, $l_2 > 0$ and $l_s = 0$ for $s = 3, 4, 5, \dots, q$. Let l be an n -weighting line on which by (i) of Lemma (3.1) there are no points of weight 0 and suppose that on l there are α points of weight 1 and β points of weight 2. Then counting points of l , gives the following:

$$\alpha + \beta = q + 1$$

And counting the weights of points on l , we get

$$\alpha + 2\beta = n$$

Solving these two equations gives

$$\alpha = 2(\square + l) - \square \quad (3.1)$$

$$\text{And} \quad \beta = \square - (\square + l) \quad (3.2)$$

Counting incidences between points of weight 2 and $\square$ -secants gives

$$\square_2 \square_{\square}^2 = \square_{\square} \square$$

Using Lemma 1.1 and equation (1.2) we get

$$\square_2 = \frac{(\square - \square)(\square - \square - l)}{2} \quad (3.3)$$

And counting incidences between points of weight 1 and n -secants gives

$$\square_l \square_{\square}^l = \square_{\square} \square$$

Hence, by using Lemma 1.1 and equation (1.1) we get

$$\square_l = (\square - \square)(2\square + 2 - \square)$$

Making use of the equations (3.4), (1.3) and (1.4) and the following relation:

$$\square_0 + \square_l + \square_2 = \square^2 + \square + l$$

It follows that,

$$\begin{aligned} 2\square^2 + 2\square + 2 - (\square - \square)(3\square + 3 - \square) - 2\square_0 &= 0; \\ 2\square^2 + 2\square + 2 + (\square - \square)(\square - 3\square - 3) - 2\square_0 &= 0; \\ \square^2 - (4\square + 3)\square + 5\square^2 + 5\square + 2 - 2\square_0 &= 0 \end{aligned} \quad (3.5)$$

If $\square_0 = \square^2 - 3\square + 3$, then from (1.5), we have:

$$\square^2 - (4\square + 3)\square + 3\square^2 + 11\square - 4 = 0$$

Thus,

$$\square = \frac{[(4\square + 3) \pm \sqrt{4\square^2 - 20\square + 25}]}{2} = \frac{[(4\square + 3) \pm (2\square - 5)]}{2}$$

Therefore, $\square = \square + 4$ or $\square = 3\square - 1$.

Lemma 3.2. The value of $\square$ in the $(k, \square; f)$ – arcs of type $(\square - \square, \square)$ in $\square\square(2, \square)$, with $\square = (\square - \square)(\square + 1)$ and $\text{Im}(f) = \{0, 1, 2\}$ satisfies the following inequality:

$$\square + 1 < \square < 2\square + 2.$$

Proof. For $\square \leq \square + 1$ we get $\square_2 \leq 0$. But $\square_2 > 0$ and $\square_2$ cannot be zero because this implies that $\text{Im}(f) = \{0, 1\}$ and this contradiction. For $\square \geq 2\square + 2$ we get $\square_1 \leq 0$. But $\square_1 > 0$ and $\square_1$ cannot be zero because this implies that $\text{Im}(f) = \{0, 2\}$ and this contradiction. $\square$

Lemma 3.3. In the $(k, \square; f)$ – arcs of type $(\square - \square, \square)$ in $\square\square(2, \square)$, if $\square = (\square - \square)(\square + 1)$, $\text{Im}(f) = \{0, 1, 2\}$ and $\square_0 = \square^2 - 3\square + 3$, then the value of $\square = \square + 4$.

Proof. If $\square = 3\square - 1$, then from Lemma (3.2), we must have $3\square - 1 < 2\square + 2$, i.e. $2\square + 2 - 3\square + 1 > 0$. But $2\square + 2 - 3\square + 1 = 3 - \square < 0$, if $\square > 3$. $\square$

Theorem 3.1. When $\text{Im}(f) = \{0, 1, 2\}$ and $\square_0 = \square^2 - 3\square + 3$, there exists $(4\square - 2, \square + 4; f)$ – arc of type $(4, \square + 4)$ in $\square\square(2, \square)$, having:

- (i) $\square = 4(\square + 1)$ and $\square_1 = 4\square - 8$;
- (ii) $\square_2 = 6$ and its coordinates points form a $(6, 3)$ – arc with four 3– secants in $\square\square(2, \square)$.

Proof. From Lemma 1.3, we have $\square = \square + 4$, and then from (3.3) and (3.4), we get:

$$\square_1 = 4\square - 8; \quad \square_2 = 6.$$

Also from Lemma 3.1, we obtain:

$$\square_{\square+4} = 4.$$

From the equations (3.1) and (3.2), we have:

$$\alpha = \square - 2; \quad \beta = 3.$$

Then the points of weight 2 form $(6, 3)$ – arc $\square$, has four 3 – secants because there are three of them on $(\square + 4)$ – weighting line and there are no more than two of them on 4– weighting line. That is $\square_3 = \square_{\square+4} = 4$ and $\square_0 + \square_1 + \square_2 = \square_4 = \square^2 + \square - 3$. Also the points of weight 1 are the remaining points on 3 – secants of $\square$. $\square$

Theorem 3.2. There exist $(\square + 1 + \frac{\square(\square+1)}{2}, 2\square + 1; \square)$ of type $(\square + 1, 2\square + 1)$ in $\square\square(2, \square)$, where the points of weight 1 formed a conic, $\square$ is minimal and $\text{Im}\square = \{0, 1, 2\}$.

Proof. If $\square = 2\square + 1$, then equations (3.3) and (3.4), give:

$$\square_2 = \frac{\square(\square+1)}{2}; \quad \square_1 = \square + 1.$$

Then the point of weight 1 are the points of the conic in $\square\square(2, \square)$, and the points of weight 2 are the exterior points of the conic. Then the points of weight 0 are the interior points of the conic. The tangent lines of the conic have weight equal $2\square + 1$, the exterior lines and the bisecant of the conic have weight equal $\square + 1$.

Hence from Lemma (3.1), part (ii), we have $\square_{2\square+1} = \square + 1$ and $\square_{\square+1} = \square^2$.

Since $\square = \square_1 + \square_2 = (\square + 1) + \frac{\square(\square+1)}{2}$.

Therefore, there exist $\left(\square + 1 + \frac{\square(\square+1)}{2}, 2\square + 1; \square\right)$ of type $(\square + 1, 2\square + 1)$ in $\square\square(2, \square)$. $\square$

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حول الأقواس الموزونة من النوع $(\square - \square, \square)$ في $\square\square(\square, \square)$

فؤاد كاظم حميد

قسم الرياضيات / كلية التربية للعلوم الصرفة / جامعة البصرة

الخلاصة:

في هذا البحث برهنا وجود اصغر قوس $(\square - \square, \square)$ من النوع $(4\square - 2, \square + 4)$ الذي له اصغر $\square$ و لا يحتوي على نقطة وزنها اكبر من 2 في المستوي الاسقاطي من الرتبة $\square$. كذلك برهنا وجود القوس $(\square + 1 + \frac{\square(\square+1)}{2}, 2\square + 1; \square)$ من النوع $(\square + 1, 2\square + 1)$ في $PG(2, \square)$ حيث النقاط التي لها وزن 1 تشكل قطع مخروطي و النقاط التي لها وزن 2 هي النقاط الخارجية لهذا القطع المخروطي و النقاط التي لها وزن 0 هي النقاط الداخلية للقطع المخروطي و $\square$ هو الاصغر و $Im(\square) = \{0, 1, 2\}$.

الكلمات المفتاحية: المستوي الاسقاطي من الرتبة $\square$ ، الاقواس الموزونة $(\square, \square)$.

تصنيف الموضوع رياضياً: 51E21 .