

Irreducible modular spin characters of the symmetric group S_{19} modulo $p = 11$

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Abstract:

The main result of this paper is to find the irreducible modular spin characters of the symmetric group S_{19} modulo $p=11$.

Section (1):

Introduction (1.1):

The symmetric group S_n has a representation group $\overline{S_n}$ of order $2(n!)$ [I.Schur 1911], this group $\overline{S_n}$ has two kinds of characters, the first kind called ordinary characters which are indexed by the partitions of n . The second kind called spin (projective) characters of S_n which are indexed by the partitions of n with distinct parts which are called bar partitions of n [A.O.Morris and A.K.Yaseen 1988]. Finding irreducible modular spin characters equivalent to calculate the decomposition matrix [A.K.Yaseen 1987]. For $p=11$ Yaseen [A.K.Yaseen 1987] was found the modular spin characters of S_n for $11 \leq n \leq 14$ and for $n = 15, 16$ also calculated by Yaseen [A.K.Yaseen 1995, see also Maas 2011], for $n = 17$ and 18 was calculate by Jassim and Taban [A.H. Jassim and S.A.Taban 2012]. In this paper we continue the works and we success to find the irreducible modular spin characters for $n = 19$ which we give in the appendix II.

Method for calculation (1.2):

There is no general method for a finding the Brauer characters for any non-abelian group, her we used several techniques as in [G.D. James and A. Kerber 1981], [A.H. Jassim and S.A.Taban 2012], [S.A.Taban 1989], [A.K.Yaseen 1987] and [A.K.Yaseen 1995]. For the theory and back ground of a modular spin (projective) representation we refer to references [C.W. Curtis and I. Reiner 1966], [L. Dornhoff 1971, 1972] and [B.M.Puttaswamaiah and J.D.Dixon 1977]. We give below some basic result which we used several times in calculations:

1. The degree of the spin characters $\langle \alpha \rangle = \langle \alpha_1, \dots, \alpha_m \rangle$ is:

$\deg\langle\alpha\rangle = 2^{\lfloor \frac{n-m}{2} \rfloor} \frac{n!}{\prod_{i=1}^m (\alpha_i!)} \prod_{1 \leq i < j \leq m} (\alpha_i - \alpha_j) / (\alpha_i + \alpha_j)$ [A.O.Morris 1962], [A.O.Morris and A.K.Yaseen 1988].

2. Let B be the block of defect one and let b be the number of p -conjugate characters to the irreducible ordinary character χ of G . Then:
 - a) There exists a positive integer number N such that the irreducible ordinary characters of G are lying in the block B divided into two disjoint classes: $B_1 = \{\chi \in B \mid b \deg \chi \equiv N \pmod{p^a}\}$, $B_2 = \{\chi \in B \mid b \deg \chi \equiv -N \pmod{p^a}\}$
 - b) Each coefficient of the decomposition matrix of the block B is 1 or 0.
 - c) If α_1 and α_2 are not p -conjugate characters and are belong to the same class (B_1, B_2) above, then they have no irreducible modular character in common.
 - d) For every irreducible ordinary character χ in B_1 , there exists irreducible ordinary character φ in B_2 such that they have one irreducible modular character in common with one multiplicity [B.M.Puttaswamaiah and J.D.Dixon 1977].
3. If C is a principal character of G for an odd prime p and all the entries in C are divisible by a non-negative integer q , then $(1 \setminus q)C$ is a principal character of G [G.D.James and A.Kerber 1981].
4. Let p be odd then:
 - a) If n odd and $p \nmid n$ then $\langle n \rangle$ is irreducible modular spin character.
 - b) If n is odd and $p \nmid n$ nor $p \nmid (n-1)$, then $\langle n-1, 1 \rangle$ and $\langle n-1, 1 \rangle'$ are distinct irreducible modular spin characters [A.K.Yaseen 1987].
5. If C is a principal character of G for a prime p , then $\deg C \equiv 0 \pmod{p^A}$, where $o(G) = p^A m$, $(p, m) = 1$ [S.A.Taban 1989], [J.F.Humphreys 1977].
6. If the decomposition matrix $D_{n-1,p} = (d_{ij})$ for S_{n-1} is known, then we can induced columns $(\psi_j \uparrow^{(r,\bar{r})} S_n)$ for S_n , these columns are a linear combination with non-negative coefficients from the columns of $D_{n,p}$ [G.D.James and A.Kerber 1981].

Notation(1.3):

p.s.	principle spin character.
p.i.s.	principle indecomposable spin character.
m.s.	modular spin character.
i.m.s.	irreducible modular spin character.
$\langle \lambda \rangle^{no}$	(no) mean the number of i.m.s. in $\langle \lambda \rangle$
$\equiv$	equivalence mod 11.

Section (2) Brauer trees to the symmetric group S_{19} , $p=11$:

The decomposition matrix for S_{19} modulo $p=11$ of degree (81,72) [A.O.Morris 1962],[A.O.Morris and A.K.Yaseen 1988]. There are 33 blocks six of them $B_1, B_2, \dots, B_6$, are of defect one and the others $B_7, \dots, B_{33}$ of defect zero.

Lemma (2.1)

The Brauer tree for the block B_1 is:

$$\langle 19 \rangle^* \text{ --- } \langle 11,8 \rangle = \langle 11,8 \rangle' \text{ --- } \langle 10,8,1 \rangle^* \text{ --- } \langle 9,8,2 \rangle^* \text{ --- } \langle 8,7,4 \rangle^* \text{ --- } \langle 8,6,5 \rangle^*$$

Proof:

On 11-regular classes we have $\langle 11,8 \rangle = \langle 11,8 \rangle'$

$$a) \deg(\langle 11,8 \rangle + \langle 11,8 \rangle') \equiv \deg \langle 9,8,2 \rangle^* \equiv \deg \langle 8,6,5 \rangle^* \equiv 5.$$

$$\deg \langle 19 \rangle^* \equiv \deg \langle 10,8,1 \rangle^* \equiv \deg \langle 8,7,4 \rangle^* \equiv -5$$

b) By using (8,4)-inducing of p.i.s D_1, D_3, D_5, D_7 and D_9 for S_{18} (see appendix I) to S_{19} we have respectively :

$$1. \langle 19 \rangle + \langle 11,8 \rangle + \langle 11,8 \rangle'$$

$$2. \langle 11,8 \rangle + \langle 11,8 \rangle' + \langle 10,8,1 \rangle^*$$

$$3. \langle 10,8,1 \rangle^* + \langle 9,8,2 \rangle^*$$

$$4. \langle 9,8,2 \rangle^* + \langle 8,7,4 \rangle^*$$

$$5. \langle 8,7,4 \rangle^* + \langle 8,6,5 \rangle^*$$

we have the Brauer tree for this block B_1 . ■

Lemma(2.2)

The Brauer tree for the block B_2 is:

$$\begin{array}{ccc} \langle 18,1 \rangle - \langle 12,7 \rangle & \backslash & \langle 11,7,1 \rangle^* / \langle 9,7,2,1 \rangle - \langle 8,7,3,1 \rangle - \langle 7,6,5,1 \rangle \\ \langle 18,1 \rangle' - \langle 12,7 \rangle' & / & \backslash \langle 9,7,2,1 \rangle' - \langle 8,7,3,1 \rangle' - \langle 7,6,5,1 \rangle' \end{array}$$

Proof:

$$a) \deg\{\langle 18,1 \rangle, \langle 18,1 \rangle', \langle 11,7,1 \rangle^*, \langle 8,7,3,1 \rangle, \langle 8,7,3,1 \rangle'\} \equiv 7$$

$$\deg\{\langle 12,7 \rangle, \langle 12,7 \rangle', \langle 9,7,2,1 \rangle, \langle 9,5,2,1 \rangle', \langle 7,6,5,1 \rangle, \langle 7,6,5,1 \rangle'\} \equiv -7.$$

b) By using (1,0)-inducing of p.i.s

$D_5, D_6, D_7, D_8, D_9, D_{10}, D_{11}, D_1, D_2$ and D_3 for S_{18} (see appendix I) to S_{19} we have p.s. $c_5, c_6, c_7, c_8, c_9, c_{10}, k_1, k_2, k_3, k_4$ respectively.

So we have the approximation matrix (Table (1))

(Table (1))

$\langle 18,1 \rangle$	1	1								
$\langle 18,1 \rangle'$	1		1							
$\langle 12,7 \rangle$	1	1	1	1						
$\langle 12,7 \rangle'$	1	1	1	1						
$\langle 11,7,1 \rangle^*$		1	1	2	1	1				
$\langle 9,7,2,1 \rangle$					1		1			
$\langle 9,7,2,1 \rangle'$						1		1		
$\langle 8,7,3,1 \rangle$							1		1	
$\langle 8,7,3,1 \rangle'$								1		1
$\langle 7,6,5,1 \rangle$									1	
$\langle 7,6,5,1 \rangle'$										1
	k_1	k_2	k_3	k_4	c_5	c_6	c_7	c_8	c_9	c_{10}

$\langle 18,1 \rangle \neq \langle 18,1 \rangle'$ so k_1 splits to c_1 and c_2 .

$k_4 = k_2 + k_3 - c_1 - c_2$, this give p.i.s. $k_2 - c_1$, $k_3 - c_2$.

Hence we have the Brauer tree for this block B_2 ■.

Lemma(2.3)

The Brauer tree for the block B_5 is:

$$\langle 16,2,1 \rangle^* _ \langle 13,5,1 \rangle^* _ \langle 12,5,2 \rangle^* _ \langle 11,5,2,1 \rangle = \langle 11,5,2,1 \rangle' _ \langle 8,5,3,2,1 \rangle^* _ \langle 7,5,4,2,1 \rangle^*$$

Proof:

on 11-regular classes we have $\langle 11,5,2,1 \rangle = \langle 11,5,2,1 \rangle'$

$$a) \deg \langle 13,5,1 \rangle^* \equiv \deg \langle 11,5,2,1 \rangle + \langle 11,5,2,1 \rangle' \equiv \deg \langle 7,5,4,2,1 \rangle^* \equiv 7$$

$$\deg \langle 16,2,1 \rangle^* \equiv \deg \langle 12,5,2 \rangle^* \equiv \deg \langle 8,5,3,2,1 \rangle^* \equiv -7$$

b) By using $(r, \bar{r})$ -inducing of p.i.s. $D_{16}, D_{18}, D_{19}, D_{20}$ and D_{28} , for S_{18} (see appendix I) to S_{19} we get respectively .

$$1) \langle 16,2,1 \rangle^* + \langle 13,5,1 \rangle^*.$$

$$2) 2\langle 12,5,2 \rangle^* + 2\langle 11,5,2,1 \rangle + 2\langle 11,3,2,1 \rangle' = 2k \text{ . sok is p.i.}$$

$$3) \langle 11,5,2,1 \rangle + \langle 11,5,2,1 \rangle' + \langle 8,5,3,2,1 \rangle^*.$$

$$4) \langle 8,5,3,2,1 \rangle^* + \langle 7,5,4,2,1 \rangle^*.$$

$$5) \langle 13,5,1 \rangle^* + \langle 12,5,2 \rangle^*.$$

So, we get the Brauer tree for the block B_5 ■.

Lemma(2.4)

The Brauer tree for the block B_6 is:

$$\langle 15,3,1 \rangle^* _ \langle 14,4,1 \rangle^* _ \langle 12,4,3 \rangle^* _ \langle 11,4,3,1 \rangle = \langle 11,4,3,1 \rangle' _ \langle 9,4,3,2,1 \rangle^* _ \langle 6,5,4,3,1 \rangle^*$$

Proof:

on 11-regular classes we have $\langle 11,4,3,1 \rangle = \langle 11,4,3,1 \rangle'$

$$a) \deg \langle 14,4,1 \rangle^* \equiv \deg \langle 11,4,3,1 \rangle + \langle 11,4,3,1 \rangle' \equiv \deg \langle 6,5,4,3,1 \rangle^* \equiv 8$$

$$\deg \langle 15,3,1 \rangle^* \equiv \deg \langle 12,4,3 \rangle^* \equiv \deg \langle 9,4,3,2,1 \rangle^* \equiv -8$$

b) By using $(r, \bar{r})$ -inducing of p.i.s. $D_{21}, D_{23}, D_{24}, D_{25}, D_{28}$, for S_{18} (see appendix I) to S_{19} we get respectively:

$$1) \langle 15,3,1 \rangle^* + \langle 14,4,1 \rangle^*.$$

$$2) 2\langle 12,4,3 \rangle^* + 2\langle 11,4,3,1 \rangle + 2\langle 11,4,3,1 \rangle' = 2l \text{ .sol is p.i.}$$

$$3) \langle 11,4,3,1 \rangle + \langle 11,4,3,1 \rangle' + \langle 9,4,3,2,1 \rangle^*.$$

$$4) \langle 9,4,3,2,1 \rangle^* + \langle 6,5,4,3,1 \rangle^*.$$

$$5) \langle 14,4,1 \rangle^* + \langle 12,4,3 \rangle^*.$$

So, we get the Brauer tree for the block B_6 ■.

Lemma(2.5)

The Brauer tree for the block B_3 is:

$$\begin{array}{ccc} \langle 17,2 \rangle - \langle 13,6 \rangle & \backslash & \langle 11,6,2 \rangle^* / \langle 10,6,2,1 \rangle - \langle 8,6,3,2 \rangle - \langle 7,6,4,2 \rangle \\ \langle 17,2 \rangle' - \langle 13,6 \rangle' & / & \backslash \langle 10,6,2,1 \rangle' - \langle 8,6,3,2 \rangle' - \langle 7,6,4,2 \rangle' \end{array}$$

Proof:

$$a) \deg \{ \langle 17,2 \rangle, \langle 17,2 \rangle', \langle 11,6,2 \rangle^*, \langle 8,6,3,2 \rangle, \langle 8,6,3,2 \rangle' \} \equiv 9$$

$$\deg \{ \langle 13,6 \rangle, \langle 13,6 \rangle', \langle 10,6,2,1 \rangle, \langle 10,6,2,1 \rangle', \langle 7,6,4,2 \rangle, \langle 7,6,4,2 \rangle' \} \equiv -9.$$

b) By using $(r, \bar{r})$ -inducing we have (see appendix I):

$$D_{11} \uparrow^{(2,10)} S_{19} = k_1, D_{12} \uparrow^{(2,10)} S_{19} = k_2, D_{14} \uparrow^{(2,10)} S_{19} = k_3$$

$$D_{15} \uparrow^{(2,10)} S_{19} = k_4, \langle 10,6,2 \rangle \uparrow^{(0,1)} S_{19} = c_5, \langle 10,6,2 \rangle' \uparrow^{(0,1)} S_{19} = c_6.$$

Thus, we have the approximation matrix (Table (2))

Table(2)

	Ψ_1	Ψ_2	φ_5	φ_6	Ψ_3	Ψ_4	φ_1	φ_2
$\langle 17,2 \rangle$	1						a	
$\langle 17,2 \rangle'$	1							a
$\langle 13,6 \rangle$	1	1					b	
$\langle 13,6 \rangle'$	1	1						b
$\langle 11,6,2 \rangle^*$		2	1	1			c	c
$\langle 10,6,2,1 \rangle$			1		1		d	
$\langle 10,6,2,1 \rangle'$				1	1			d
$\langle 8,6,3,2 \rangle$					1	1	f	
$\langle 8,6,3,2 \rangle'$					1	1		f
$\langle 7,6,4,2 \rangle$						1	h	
$\langle 7,6,4,2 \rangle'$						1		h
	k_1	k_2	c_5	c_6	k_3	k_4	Y_1	Y_2

Since $\langle 17,2 \rangle \neq \langle 17,2 \rangle'$ on $(11, \alpha)$ -regular classes then either k_1 is split or there are another two columns. Suppose there are two columns such as Y_1 and Y_2

To describe columns Y_1 and Y_2

1. $\langle 17,2 \rangle \downarrow S_{18} = (\langle 16,2 \rangle^*)^1 + (\langle 17,1 \rangle^*)^1$ has 2 of i.m.s (see appendix I) and form (Table(2)) we have $a \in \{0,1\}$.
2. $\langle 13,6 \rangle \downarrow S_{18} = (\langle 12,6 \rangle^*)^2 + (\langle 13,5 \rangle^*)^2$ has 4 of i.m.s. we have $b \in \{0,1,2\}$.
3. $\langle 11,6,2 \rangle^* \downarrow S_{18} = (\langle 10,6,2 \rangle)^1 + (\langle 10,6,2 \rangle')^1 + (\langle 11,5,2 \rangle)^2 + (\langle 11,5,2 \rangle')^2 + (\langle 11,6,1 \rangle)^2 + (\langle 11,6,1 \rangle')^2$ has 10 of i.m.s. we have $c \in \{0,1,2,3,4,5,6\}$.
4. $\langle 10,6,2,1 \rangle \downarrow S_{18} = (\langle 9,6,2,1 \rangle^*)^2 + (\langle 10,5,2,1 \rangle^*)^2 + (\langle 10,6,2 \rangle)^1$ has 5 of i.m.s. we have $d \in \{0,1,2,3\}$.
5. $\langle 8,6,3,2 \rangle \downarrow S_{18} = (\langle 7,6,3,2 \rangle^*)^1 + (\langle 8,5,3,2 \rangle^*)^2 + (\langle 8,6,3,1 \rangle^*)^2$ has 5 of i.m.s. we have $f \in \{0,1,2,3\}$
6. $\langle 7,6,4,2 \rangle \downarrow S_{18} = (\langle 7,5,4,2 \rangle^*)^1 + (\langle 7,6,3,2 \rangle^*)^1 + (\langle 7,6,4,1 \rangle^*)^1$ has 3 of i.m.s. we have $h \in \{0,1,2\}$.

If $a = 0$ then k_1 splits to give $\langle 17,2 \rangle + \langle 13,6 \rangle$ and $\langle 17,2 \rangle' + \langle 13,6 \rangle'$

If $a = 1$:

- 1) Since $\langle 17,2 \rangle \downarrow S_{18} \cap \langle 13,6 \rangle \downarrow S_{18} = 2$ of i.m.s for S_{18} , then
 $\langle 17,2 \rangle \cap \langle 13,6 \rangle = \Psi_1 + \varphi_1$ if $b \in \{1,2\}$
 $= \varphi_1$ if $b = 0$;

- 2) There is no i.m.s. in $\langle 17,2 \rangle \downarrow S_{18} \cap \langle 11,6,2 \rangle^* \downarrow S_{18}, \text{soc} = 0$;
- 3) There is no i.m.s. in $\langle 17,2 \rangle \downarrow S_{18} \cap \langle 10,6,2,1 \rangle \downarrow S_{18}, \text{sod} = 0$;
- 4) There is no i.m.s. in $\langle 17,2 \rangle \downarrow S_{18} \cap \langle 8,6,3,2 \rangle \downarrow S_{18}, \text{sof} = 0$;
- 5) There is no i.m.s. in $\langle 17,2 \rangle \downarrow S_{18} \cap \langle 7,6,4,2 \rangle \downarrow S_{18}, \text{soh} = 0$.

We have the possible columns

$$Y_1 = \langle 17,2 \rangle + b \langle 13,6 \rangle ,$$

$$Y_2 = \langle 17,2 \rangle' + b \langle 13,6 \rangle'; b \in \{0,1,2\}$$

$$\deg Y_1 \equiv 0 \text{ and } \deg Y_2 \equiv 0 \text{ only when } b = 1$$

So k_1 splits to $\langle 17,2 \rangle + \langle 13,6 \rangle$, and $\langle 17,2 \rangle' + \langle 13,6 \rangle'$

Since $\langle 13,6 \rangle \neq \langle 13,6 \rangle'$ on $(11, \alpha)$ -regular classes then either k_2 splits or there are two columns. If we suppose there are another two columns such as Y_1 and Y_2 (as in Table (2)); to describe these two columns:

- 1) Since $\langle 13,6 \rangle \downarrow S_{18} \cap \langle 11,6,2 \rangle^* \downarrow S_{18}$ has 2 of i.m.s for S_{18} ,
 $\langle 13,6 \rangle \cap \langle 11,6,2 \rangle^* = \Psi_2 + \varphi_3$ if $c \in \{1,2,3,4,5,6\}$ and $b = 1$ or $c = 1$ and $b \in \{1,2\}$
 $= \Psi_2 + 2\varphi_3$ if $c \in \{2,3,4,5,6\}$ and $b = 2$
 $= \Psi_2$ if $c = 0$;
- 2) There is no i.m.s. in $\langle 13,6 \rangle \downarrow S_{18} \cap \langle 10,6,2,1 \rangle \downarrow S_{18}, \text{sod} = 0$;
- 3) There is no i.m.s. in $\langle 13,6 \rangle \downarrow S_{18} \cap \langle 8,6,3,2 \rangle \downarrow S_{18}, \text{sof} = 0$;
- 4) There is no i.m.s. in $\langle 13,6 \rangle \downarrow S_{18} \cap \langle 7,6,4,2 \rangle \downarrow S_{18}, \text{soh} = 0$.

We get the possible columns

$$Y_1 = b \langle 13,6 \rangle + c \langle 11,6,2 \rangle^* ,$$

$$Y_2 = b \langle 13,6 \rangle' + c \langle 11,6,2 \rangle^* , b \in \{1,2\}, c \in \{0,1,2,3,4,5,6\}$$

$$\deg Y_1 \equiv 0 \text{ and } \deg Y_2 \equiv 0 \text{ only when } b = c.$$

So k_2 splits to give $\langle 13,6 \rangle + \langle 11,6,2 \rangle^*$ and $\langle 13,6 \rangle' + \langle 11,6,2 \rangle^*$ which is the same when $b = 0$.

Since $\langle 7,6,4,2 \rangle \neq \langle 7,6,4,2 \rangle'$ on $(11, \alpha)$ -regular classes then k_4 splits or there are two columns. If we suppose there are two columns such as Y_1 and Y_2 (as in Table (2)). To describe Y_1 and Y_2 :

If $h \in \{1,2\}$:

- 1) There is no i.m.s. in $\langle 7,6,4,2 \rangle \downarrow S_{18} \cap \langle 11,6,2 \rangle^* \downarrow S_{18}, \text{soc} = 0$;
- 2) There is no i.m.s. in $\langle 7,6,4,2 \rangle \downarrow S_{18} \cap \langle 10,6,2,1 \rangle \downarrow S_{18}, \text{sod} = 0$;
- 3) Since $\langle 7,6,4,2 \rangle \downarrow S_{18} \cap \langle 8,6,3,2 \rangle \downarrow S_{18}$ has 2 of i.m.s for S_{18}
 $\langle 7,6,4,2 \rangle \cap \langle 8,6,3,2 \rangle = \Psi_4 + \varphi_9$ if $f \in \{1,2,3\}$ and $h = 1$ or $f = 1$ and $h \in \{1,2\}$
 $= \Psi_4 + 2\varphi_9$ if $f \in \{2,3\}$ and $h = 2$
 $= \Psi_4$ if $f = 0$.

We get the possible columns

$$Y_1 = f \langle 8,6,3,2 \rangle + \langle 7,6,4,2 \rangle,$$

$$Y_2 = f \langle 8,6,3,2 \rangle' + \langle 7,6,4,2 \rangle' , f \in \{0,1,2,3\}, h \in \{1,2\}$$

$$\deg Y_1 \equiv 0 \text{ and } \deg Y_2 \equiv 0 \text{ only when } f = h.$$

So, k_4 splits to $\langle 8,6,3,2 \rangle + \langle 7,6,4,2 \rangle$ and $\langle 8,6,3,2 \rangle' + \langle 7,6,4,2 \rangle'$ which is the same when $h = 0$.

Now, since $\langle 8,6,3,2 \rangle \neq \langle 8,6,3,2 \rangle'$ on $(11, \alpha)$ -regular classes, then k_3 splits or there are two columns. If we suppose there are another two columns such as Y_1 and Y_2 (as in Table (2)); to describe Y_1 and Y_2 :

If $f \in \{1,2,3\}$:

1) There is no i.m.s. in $\langle 8,6,3,2 \rangle \downarrow S_{18} \cap \langle 11,6,2 \rangle^* \downarrow S_{18}$, $\text{soc} = 0$;

2) Since $\langle 8,6,3,2 \rangle \downarrow S_{18} \cap \langle 10,6,3,1 \rangle \downarrow S_{18}$ has 2 of i.m.s for S_{18}

$$\langle 8,6,3,2 \rangle \cap \langle 10,6,3,1 \rangle = \Psi_3 + \varphi_7 \text{ if } d \in \{1,2,3\} \text{ and } f = 1 \text{ or } d = 1 \text{ and } f \in \{1,2,3\}$$

$$= \Psi_3 + 2\varphi_7 \text{ if } d \in \{2,3\} \text{ and } f = 2$$

$$= \Psi_3 + 3\varphi_7 \text{ if } d = 3 \text{ and } f = 3$$

$$= \Psi_3 \quad \text{if } d = 0.$$

We get the possible columns

$$Y_1 = d \langle 10,6,2,1 \rangle + f \langle 8,6,3,2 \rangle,$$

$$Y_2 = d \langle 10,6,2,1 \rangle' + f \langle 8,6,3,2 \rangle' , d \in \{0,1,2,3\}, f \in \{1,2,3\}$$

$$\deg Y_1 \equiv 0 \text{ and } \deg Y_2 \equiv 0 \text{ only when } d = f.$$

So, k_4 splits to $\langle 10,6,2,1 \rangle + \langle 8,6,3,2 \rangle$ and $\langle 10,6,2,1 \rangle' + \langle 8,6,3,2 \rangle'$ which is the same when $f = 0$. So we get the Brauer tree for the block B_3 ■.

Theorem

The decomposition matrix of the spin characters for S_{19} modulo $p = 11$ as given in appendix II

Proof:

We determine the decomposition matrices for all blocks of S_{19} except the block B_4 to complete the proof we calculate the decomposition matrix for the block B_4 .

$$\text{a) } \deg\{\langle 16,3 \rangle, \langle 16,3 \rangle', \langle 11,5,3 \rangle^*, \langle 9,5,3,2 \rangle, \langle 9,5,3,2 \rangle'\} \equiv 9$$

$$\deg\{\langle 14,5 \rangle, \langle 14,5 \rangle', \langle 10,5,3,1 \rangle, \langle 10,5,3,1 \rangle', \langle 7,5,4,3 \rangle, \langle 7,5,4,3 \rangle'\} \equiv -9.$$

b) By using $(r, \bar{r})$ -inducing we get:

$$D_{16} \uparrow^{(3,9)} S_{19} = k_1, D_{17} \uparrow^{(3,9)} S_{19} = k_2, D_{19} \uparrow^{(3,9)} S_{19} = k_3$$

$$D_{20} \uparrow^{(3,9)} S_{19} = k_4, \langle 10,5,3 \rangle \uparrow^{(0,1)} S_{19} = c_5, \langle 10,5,3 \rangle' \uparrow^{(0,1)} S_{19} = c_6.$$

Thus, we have the approximation matrix (Table (3))

Table(3)

	Ψ_1	Ψ_2	φ_5	φ_6	Ψ_3	Ψ_4	φ_1	φ_2
$\langle 16,3 \rangle$	1						a	
$\langle 16,3 \rangle'$	1							a
$\langle 14,5 \rangle$	1	1					b	
$\langle 14,5 \rangle'$	1	1						b
$\langle 11,5,3 \rangle^*$		2	1	1			c	c
$\langle 10,5,3,1 \rangle$			1		1		d	
$\langle 10,5,3,1 \rangle'$				1	1			d
$\langle 9,5,3,2 \rangle$					1	1	f	
$\langle 9,5,3,2 \rangle'$					1	1		f
$\langle 7,5,4,3 \rangle$						1	h	
$\langle 7,5,4,3 \rangle'$						1		h
	k_1	k_2	c_5	c_6	k_3	k_4	Y_1	Y_2

Since $\langle 16,3 \rangle \neq \langle 16,3 \rangle'$ on $(11, \alpha)$ -regular classes then either k_1 is split or there are two columns.

Suppose there are two columns such as Y_1 and Y_2 ; to describe columns Y_1 and Y_2

1. $\langle 16,3 \rangle \downarrow S_{18} = (\langle 15,3 \rangle^*)^1 + (\langle 16,2 \rangle^*)^1$ has 2 of i.m.s (see appendix I) and from (Table(2)) we have $a \in \{0,1\}$.
2. $\langle 14,5 \rangle \downarrow S_{18} = (\langle 13,5 \rangle^*)^2 + (\langle 14,4 \rangle^*)^2$ has 4 of i.m.s. we have $b \in \{0,1,2\}$.
3. $\langle 11,5,3 \rangle^* \downarrow S_{18} = (\langle 10,5,3 \rangle)^1 + (\langle 10,5,3 \rangle')^1 + (\langle 11,4,3 \rangle)^2 + (\langle 11,4,3 \rangle')^2 + (\langle 11,5,2 \rangle)^2 + (\langle 11,5,2 \rangle')^2$ has 10 of i.m.s. we have $c \in \{0,1,2,3,4,5,6\}$.
4. $\langle 10,5,3,1 \rangle \downarrow S_{18} = (\langle 9,5,3,1 \rangle^*)^1 + (\langle 10,4,3,1 \rangle^*)^2 + (\langle 10,5,2,1 \rangle)^2 + (\langle 10,5,3 \rangle)^1$ has 6 of i.m.s. we have $d \in \{0,1,2,3,4\}$.
5. $\langle 9,5,3,2 \rangle \downarrow S_{18} = (\langle 8,5,3,2 \rangle^*)^2 + (\langle 9,4,3,2 \rangle^*)^2 + (\langle 9,5,3,1 \rangle)^1$ has 5 of i.m.s. we have $f \in \{0,1,2,3\}$.
6. $\langle 7,5,4,3 \rangle \downarrow S_{18} = (\langle 6,5,4,3 \rangle^*)^1 + (\langle 7,5,4,2 \rangle^*)^1$ has 2 of i.m.s. we have $h \in \{0,1\}$.

If $a = 0$ then k_1 splits to give $\langle 16,3 \rangle + \langle 14,5 \rangle$ and $\langle 16,3 \rangle' + \langle 14,5 \rangle'$

If $a = 1$:

- 1) Since $\langle 16,3 \rangle \downarrow S_{18} \cap \langle 14,5 \rangle \downarrow S_{18}$ has 2 of i.m.s for S_{18}
 $\langle 16,3 \rangle \cap \langle 14,5 \rangle = \Psi_1 + \varphi_1$ if $b \in \{1,2\}$
 $= \varphi_1$ if $b = 0$;
- 2) There is no i.m.s. in $\langle 16,3 \rangle \downarrow S_{18} \cap \langle 11,5,3 \rangle^* \downarrow S_{18}, \text{soc} = 0$;

- 3) There is no i.m.s. in $\langle 16,3 \rangle \downarrow S_{18} \cap \langle 10,5,3,1 \rangle \downarrow S_{18}$ so $d = 0$;
- 4) There is no i.m.s. in $\langle 16,3 \rangle \downarrow S_{18} \cap \langle 9,5,3,2 \rangle \downarrow S_{18}$, so $f = 0$;
- 5) There is no i.m.s. in $\langle 16,3 \rangle \downarrow S_{18} \cap \langle 7,5,4,3 \rangle \downarrow S_{18}$, so $h = 0$.

Now, we get the possible columns

$$Y_1 = \langle 16,3 \rangle + b \langle 14,5 \rangle ,$$

$$Y_2 = \langle 16,3 \rangle' + b \langle 14,5 \rangle' , b \in \{0,1,2\}$$

$$\deg Y_1 \equiv 0 \text{ and } \deg Y_2 \equiv 0 \text{ only when } b = 1$$

So k_1 splits to $\langle 16,3 \rangle + \langle 14,5 \rangle$, and $\langle 16,3 \rangle' + \langle 14,5 \rangle'$

Since $\langle 14,5 \rangle \neq \langle 14,5 \rangle'$ on $(11, \alpha)$ -regular classes then either k_2 is splits or there are two columns . If we suppose there are two columns such as Y_1 and Y_2 (as in Table (3)); to describe these two columns:

If $b \in \{1,2\}$:

- 1) Since $\langle 14,5 \rangle \downarrow S_{18} \cap \langle 11,5,3 \rangle^* \downarrow S_{18}$ has 2 of i.m.s for S_{18}
 $\langle 14,5 \rangle \cap \langle 11,4,2 \rangle^* = \Psi_2 + \varphi_3$ if $c \in \{1,2,3,4,5,6\}$ and $b = 1$
or if $c = 1$ and $b \in \{1,2\}$
 $= \Psi_2 + 2\varphi_3$ if $c \in \{2,3,4,5,6\}$ and $b = 2$
 $= \Psi_2$ if $c = 0$;
- 2) There is no i.m.s. in $\langle 14,5 \rangle \downarrow S_{18} \cap \langle 10,5,3,1 \rangle \downarrow S_{18}$, so $d = 0$;
- 3) There is no i.m.s. in $\langle 14,5 \rangle \downarrow S_{18} \cap \langle 9,5,3,2 \rangle \downarrow S_{18}$, so $f = 0$;
- 4) There is no i.m.s. in $\langle 14,5 \rangle \downarrow S_{18} \cap \langle 7,5,4,3 \rangle \downarrow S_{18}$, so $h = 0$.

We get the possible columns

$$Y_1 = \langle 14,5 \rangle + c \langle 11,5,3 \rangle^* ,$$

$$Y_2 = \langle 14,5 \rangle + c \langle 11,5,3 \rangle^* , c \in \{0,1,2,3,4,5,6\}$$

$$\deg Y_1 \equiv 0 \text{ and } \deg Y_2 \equiv 0 \text{ only when } b = c$$

So k_2 splits to give $\langle 14,5 \rangle + \langle 11,5,3 \rangle^*$ and $\langle 14,5 \rangle' + \langle 11,5,3 \rangle^*$ which is the same when $b = 0$.

Since $\langle 7,5,4,3 \rangle \neq \langle 7,5,4,3 \rangle'$ on $(11, \alpha)$ -regular classes then k_4 splits or there are two columns. If we suppose there are two columns such as Y_1 and Y_2 (as in Table (3)); to describe Y_1 and Y_2 :

If $h = 1$:

- 1) There is no i.m.s. in $\langle 7,5,4,3 \rangle \downarrow S_{18} \cap \langle 11,5,3 \rangle^* \downarrow S_{18}$, so $c = 0$;
- 2) There is no i.m.s. in $\langle 7,5,4,3 \rangle \downarrow S_{18} \cap \langle 10,5,3,1 \rangle \downarrow S_{18}$, so $d = 0$;
- 3) Since $\langle 7,5,4,3 \rangle \downarrow S_{18} \cap \langle 9,5,3,2 \rangle \downarrow S_{18}$ has 2 of i.m.s for S_{18}
 $\langle 7,5,4,3 \rangle \cap \langle 9,5,3,2 \rangle = \Psi_4 + \varphi_9$ if $f \in \{1,2,3\}$
 $= \Psi_4$ if $f = 0$.

We get the possible columns

$$Y_1 = f \langle 9,5,3,2 \rangle + \langle 7,5,4,3 \rangle,$$

$$Y_2 = f \langle 9,5,3,2 \rangle' + \langle 7,5,4,3 \rangle', \quad f \in \{0,1,2,3\}$$

$$\deg Y_1 \equiv 0 \text{ and } \deg Y_2 \equiv 0 \text{ only when } f = 1$$

So, k_4 splits to $\langle 9,5,3,2 \rangle + \langle 7,5,4,3 \rangle$ and $\langle 9,5,3,2 \rangle' + \langle 7,5,4,3 \rangle'$ which is the same when $h = 0$.

Now, since $\langle 9,5,3,2 \rangle \neq \langle 9,5,3,2 \rangle'$ on $(11, \alpha)$ -regular classes and we have 9 columns, then k_3 must be a split to $\langle 10,5,3,1 \rangle + \langle 9,5,3,2 \rangle$ and $\langle 10,5,3,1 \rangle' + \langle 9,5,3,2 \rangle'$. So we get the decomposition matrix for the block B_4 ■.

From lemmas and theorem above we can find the 11-decomposition matrix for the spin characters of S_{19} . We write this decomposition matrix in appendix II

Appendix I

The decomposition matrix for the spin characters of S_{18} , $p = 11$ [A. H. Jassim and S.A. Taban]

The spin characters	The decomposition matrix for the block B_1									
$\langle 18 \rangle$	1									
$\langle 18 \rangle'$		1								
$\langle 11,7 \rangle^*$	1	1	1	1						
$\langle 10,7,1 \rangle$			1		1					
$\langle 10,7,1 \rangle'$				1		1				
$\langle 9,7,2 \rangle$					1		1			
$\langle 9,7,2 \rangle'$						1		1		
$\langle 8,7,3 \rangle$							1		1	
$\langle 8,7,3 \rangle'$								1		1
$\langle 7,6,5 \rangle$									1	
$\langle 7,6,5 \rangle'$										1
	D_1	D_2	D_3	D_4	D_5	D_6	D_7	D_8	D_9	D_{10}

The spin characters	The decomposition matrix for the block B_2				
$\langle 17,1 \rangle^*$	1				
$\langle 12,6 \rangle^*$	1	1			
$\langle 11,6,1 \rangle$		1	1		
$\langle 11,6,1 \rangle'$		1	1		
$\langle 9,6,2,1 \rangle^*$			1	1	
$\langle 8,6,3,1 \rangle^*$				1	1
$\langle 7,6,4,1 \rangle^*$					1
	D_{11}	D_{12}	D_{13}	D_{14}	D_{15}

The spin characters	The decomposition matrix for the block B_3				
$\langle 16,2 \rangle^*$	1				
$\langle 13,5 \rangle^*$	1	1			
$\langle 11,5,2 \rangle$		1	1		
$\langle 11,5,2 \rangle'$		1	1		
$\langle 10,5,2,1 \rangle^*$			1	1	
$\langle 8,5,3,2 \rangle^*$				1	1
$\langle 7,5,4,2 \rangle^*$					1
	D_{16}	D_{17}	D_{18}	D_{19}	D_{20}

The spin characters	The decomposition matrix for the block B_4				
$\langle 15,3 \rangle^*$	1				
$\langle 14,4 \rangle^*$	1	1			
$\langle 11,4,3 \rangle$		1	1		
$\langle 11,4,3 \rangle'$		1	1		
$\langle 10,4,3,1 \rangle^*$			1	1	
$\langle 9,4,3,2 \rangle^*$				1	1
$\langle 6,5,4,3 \rangle^*$					1
	D_{21}	D_{22}	D_{23}	D_{24}	D_{25}

The spin characters	The decomposition matrix for the block B_5									
$\langle 15,2,1 \rangle$	1									
$\langle 15,2,1 \rangle'$		1								
$\langle 13,4,1 \rangle$	1		1							
$\langle 13,4,1 \rangle'$		1		1						
$\langle 12,4,2 \rangle$			1		1					
$\langle 12,4,2 \rangle'$				1		1				
$\langle 11,4,2,1 \rangle^*$					1	1	1	1		
$\langle 8,4,3,2,1 \rangle$							1		1	
$\langle 8,4,3,2,1 \rangle'$								1		1
$\langle 6,5,4,2,1 \rangle$									1	
$\langle 6,5,4,2,1 \rangle'$										1
	D_{26}	D_{27}	D_{28}	D_{29}	D_{30}	D_{31}	D_{32}	D_{33}	D_{34}	D_{35}

Appendix II

The decomposition matrix for the spin characters of S_{19} , $p = 11$

The spincharacters	The decomposition matrix for the block B_1				
$\langle 19 \rangle^*$	1				
$\langle 11,8 \rangle$	1	1			
$\langle 11,8 \rangle'$	1	1			
$\langle 10,8,1 \rangle^*$		1	1		
$\langle 9,8,2 \rangle^*$			1	1	
$\langle 8,7,4 \rangle^*$				1	1
$\langle 8,6,5 \rangle^*$					1
	d_1	d_2	d_3	d_4	d_5

The spin characters	The decomposition matrix for the block B_2									
$\langle 18,1 \rangle$	1									
$\langle 18,1 \rangle'$		1								
$\langle 12,7 \rangle$	1		1							
$\langle 12,7 \rangle'$		1		1						
$\langle 11,7,1 \rangle^*$			1	1	1	1				
$\langle 9,7,2,1 \rangle$					1		1			
$\langle 9,7,2,1 \rangle'$						1		1		
$\langle 8,7,3,1 \rangle$							1		1	
$\langle 8,7,3,1 \rangle'$								1		1
$\langle 7,6,5,1 \rangle$									1	
$\langle 7,6,5,1 \rangle'$										1
	d_6	d_7	d_8	d_9	d_{10}	d_{11}	d_{12}	d_{13}	d_{14}	d_{15}

The spin characters	The decomposition matrix for the block B_3									
$\langle 17,2 \rangle$	1									
$\langle 17,2 \rangle'$		1								
$\langle 13,6 \rangle$	1		1							
$\langle 13,6 \rangle'$		1		1						
$\langle 11,6,2 \rangle^*$			1	1	1	1				
$\langle 10,6,2,1 \rangle$					1		1			

$\langle 10,6,2,1 \rangle'$						1		1		
$\langle 8,6,3,2 \rangle$							1		1	
$\langle 8,6,3,2 \rangle'$								1		1
$\langle 7,6,4,2 \rangle$									1	
$\langle 7,6,4,2 \rangle'$										1
	d_{16}	d_{17}	d_{18}	d_{19}	d_{20}	d_{21}	d_{22}	d_{23}	d_{24}	d_{25}

The spin characters	The decomposition matrix for the block B_4									
$\langle 16,3 \rangle$	1									
$\langle 16,3 \rangle'$		1								
$\langle 14,5 \rangle$	1		1							
$\langle 14,5 \rangle'$		1		1						
$\langle 11,5,3 \rangle^*$			1	1	1	1				
$\langle 10,5,3,1 \rangle$					1		1			
$\langle 10,5,3,1 \rangle'$						1		1		
$\langle 9,5,3,2 \rangle$							1		1	
$\langle 9,5,3,2 \rangle'$								1		1
$\langle 7,5,4,3 \rangle$									1	
$\langle 7,5,4,3 \rangle'$										1
	d_{26}	d_{27}	d_{28}	d_{29}	d_{30}	d_{31}	d_{32}	d_{33}	d_{34}	d_{35}

The spin characters	The decomposition matrix for the block B_5				
$\langle 16,2,1 \rangle^*$	1				
$\langle 13,5,1 \rangle^*$	1	1			
$\langle 12,5,2 \rangle^*$		1	1		
$\langle 11,5,2,1 \rangle$			1	1	
$\langle 11,5,2,1 \rangle'$			1	1	
$\langle 8,5,3,2,1 \rangle^*$				1	1
$\langle 7,5,4,2,1 \rangle^*$					1
	d_{36}	d_{37}	d_{38}	d_{39}	d_{40}

The spin characters	The decomposition matrix for the block B_6				
$\langle 15,3,1 \rangle^*$	1				
$\langle 14,4,1 \rangle^*$	1	1			
$\langle 12,4,3 \rangle^*$		1	1		
$\langle 11,4,3,1 \rangle$			1	1	
$\langle 11,4,3,1 \rangle'$			1	1	
$\langle 9,4,3,2,1 \rangle^*$				1	1
$\langle 6,5,4,3,1 \rangle^*$					1
	d_{41}	d_{42}	d_{43}	d_{44}	d_{45}

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