

Solving the equation $x^n = a$ in the Mathieu groups M_{11} , M_{12}

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Abstract

In this paper, we solve the equation $x^n = a$, in the Mathieu groups M_{11} and M_{12} where $a \in M_{11}$ or M_{12} , n is a positive integer.

Key words : Mathieu groups, finite groups, Equation in finite group, conjugate classes.

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1. Introduction

Let G be a finite group and n positive integer, the equation in G is of the form $x^n = a$, where $a \in G$ and n positive integer.

In 1903 Frobenius⁽¹⁾, Burnside⁽²⁾, advanced a computation theorem of solution for the equation $x^n = e$, e is the identity element in G . There are two ways to study the equation $x^n = a$:

1. Find the exact or approximate number of solutions for this equation. For some types of groups there are many researchers such as Frobenius⁽¹⁾, Kulakoff⁽³⁾, Huppert⁽⁴⁾, P. Hall⁽⁵⁾, Solomon⁽⁶⁾, Lam⁽⁷⁾, Yoshida⁽⁸⁾, Chowla, Herstein and Scott⁽⁹⁾, James and Kerber⁽¹⁰⁾.

2. Find the number of solutions and solutions for this equation, this study by S. Taban⁽¹¹⁾ for symmetric group, S. M. Kholil⁽¹²⁾ for Alternating group. In this paper we find the number of solutions and solutions for the equation $x^n = a$ in the Mathieu groups M_{11} , M_{12} , where

M_{11} is generated by John H., Noam D. Elkies⁽¹³⁾, A. Klymenko⁽¹⁴⁾:

$$X_1 = (1,2)(3,4)(5,6)(7,8)$$

$$X_2 = (1,2)(5,7)(6,8)(9,10)$$

$$X_3 = (1,3)(2,4)(5,6)(9,10)$$

$$X_4 = (1,3)(5,9)(6,10)(7,8)$$

$$X_5 = (1,6)(2,5)(7,8)(9,11)$$

And the Mathieu groups M_{12} is generated by:

$$X_1 = (1,6,4,2,11,3,8,9,10,7,5)$$

$$X_2 = (1,12)(2,9)(3,4)(5,6)(7,8)(10,11)$$

$$X_3 = (4,5)(2,11)(3,7)(8,9)$$

Some result of S. Taban⁽¹¹⁾ are used such as:

Lemma(1.1):

If $a \in [n]$, where $[n]$ is a conjugate class in S_n , and p prime number then:

1. $a^p \in [n]$ if p dose not divide n .
2. $a^p \in [r^p]$ if $n = rp$.

Lemma(1.2):

If $a \in [nm]$, and p is prime then :

1. $a^p \in [nm]$ if p dose not divide n and m .

2. $a^p \in [r^p n]$ if p dose not divide n and $m = pr$.

3. $a^p \in [r^p s^p]$ if $n = sp$ and $m = rp$.

Lemma(1.3):

If $a \in [n^m]$ then:

1. $a^p \in [n^m]$ if p dose not divide n .

2. $a^p \in [r^{pm}]$ if $n = rp$.

Lemma(1.4):

If $a \in [n]$, p and q are different primes then:

1. $a^{pq} \in [n]$ if p and q do not divide n .

2. $a^{pq} \in [r^p]$ if $n = rp$ and q does not divide n .

3. $a^{pq} \in [r^{pq}]$ if $n = rpq$.

Also, we are using the program Gap

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<http://www.gap-system.org/Releases/index.htm>

2. Solutions in the group M_{11}

In this Section, we find the powers C_j^n and roots, where C_j is conjugate class in M_{11} , the conjugate classes of M_{11} are introduced by G. D. James⁽¹⁵⁾ as follows:

$$C_0 = [1^{11}]$$

$$C_1 = [1^3, 2^4]$$

$$C_2 = [1^3, 4^2]$$

$$C_3 = [1, 2, 8]$$

$$C_4 = [1, 2, 8]$$

$$C_5 = [11]$$

$$C_6 = [11]$$

$$C_7 = [1, 5^2]$$

$$C_8 = [1^2, 3^3]$$

$$C_9 = [2, 3, 6]$$

Let $a \in C_3$, where

$a = (2,3)(4,5,7,6,11,9,10,8)$, we find the powers of a :

$$a^1 = (2,3)(4,5,7,6,11,9,10,8) \in C_3$$

$$a^2 = (4,7,11,10)(5,6,9,8) \in C_2$$

$$a^3 = (2,3)(4,6,10,5,11,8,7,9) \in C_3$$

$$a^4 = (4,11)(5,9)(6,8)(7,10) \in C_1$$

$$a^5 = (2,3)(4,9,7,8,11,5,10,6) \in C_4$$

$$a^6 = (4,10,11,7)(5,8,9,6) \in C_2$$

$$a^7 = (2,3)(4,8,10,9,11,6,7,5) \in C_4$$

$$a^8 = e \in C_0.$$

Let $a \in C_4$, where

$a = (2,3)(4,8,10,9,11,6,7,5)$, we find the powers of a :

$$a^1 = (2,3)(4,8,10,9,11,6,7,5) \in C_4$$

$$a^2 = (4,10,11,7)(5,8,9,6) \in C_2$$

$$a^3 = (2,3)(4,9,7,8,11,5,10,6) \in C_4$$

$$a^4 = (4,11)(5,9)(6,8)(7,10) \in C_1$$

$$a^5 = (2,3)(4,6,10,5,11,8,7,9) \in C_3$$

$$a^6 = (4,7,11,10)(5,6,9,8) \in C_2$$

$$a^7 = (2,3)(4,5,7,6,11,9,10,8) \in C_3$$

$$a^8 = e \in C_0.$$

Let $a \in C_5$, where
 $a = (1,2,3,4,9,7,10,5,6,11,8)$, we find
the powers of a :

$$a^1 = (1,2,3,4,9,7,10,5,6,11,8) \in C_5$$

$$a^2 = (1,3,9,10,6,8,2,4,7,5,11) \in C_6$$

$$a^3 = (1,4,10,11,2,9,5,8,3,7,6) \in C_5$$

$$a^4 = (1,9,6,2,7,11,3,10,8,4,5) \in C_5$$

$$a^5 = (1,7,8,9,11,4,6,3,5,2,10) \in C_5$$

$$a^6 = (1,10,2,5,3,6,4,11,9,8,7) \in C_6$$

$$a^7 = (1,5,4,8,10,3,11,7,2,6,9) \in C_6$$

$$a^8 = (1,6,7,3,8,5,9,2,11,10,4) \in C_6$$

$$a^9 = (1,11,5,7,4,2,8,6,10,9,3) \in C_5$$

$$a^{10} = (1,8,11,6,5,10,7,9,4,3,2) \in C_6$$

$$a^{11} = e$$

Let $a \in C_6$, where
 $a = (1,2,3,4,7,11,6,5,9,8,10)$, we find
the powers of a :

$$a^1 = (1,2,3,4,7,11,6,5,9,8,10) \in C_6$$

$$a^2 = (1,3,7,6,9,10,2,4,11,5,8) \in C_5$$

$$a^3 = (1,4,6,8,2,7,5,10,3,11,9) \in C_6$$

$$a^4 = (1,7,9,2,11,8,3,6,10,4,5) \in C_6$$

$$a^5 = (1,11,10,7,8,4,9,3,5,2,6) \in C_6$$

$$a^6 = (1,6,2,5,3,9,4,8,7,10,11) \in C_5$$

$$a^7 = (1,5,4,10,6,3,8,11,2,9,7) \in C_5$$

$$a^8 = (1,9,11,3,10,5,7,2,8,6,4) \in C_5$$

$$a^9 = (1,8,5,11,4,2,10,9,6,7,3) \in C_6$$

$$a^{10} = (1,10,8,9,5,6,11,7,4,3,2) \in C_5$$

$$a^{11} = e$$

The other conjugate classes can be found by result in S. A. Taban⁽¹¹⁾ as in mentioned in the introduction.

From the powers of the conjugate classes for M_{11} (table 1), we can find The roots of M_{11} as in (table 2):

3. Solutions in the group M_{12}

In this Section we find the powers C_j^n and roots, where C_j is conjugate class in M_{12} , the conjugate classes of M_{12} are introduced by G. D. James⁽¹⁵⁾ as follows:

$$C_0 = [1^{12}]$$

$$C_1 = [1^4, 2^4]$$

$$C_2 = [2^6]$$

$$C_3 = [1^4, 4^2]$$

$$C_4 = [2^2, 4^2]$$

$$C_5 = [1^2, 2, 8]$$

$$C_6 = [4, 8]$$

$$C_7 = [1^3, 3^3]$$

$$C_8 = [3^4]$$

$$C_9 = [1, 2, 3, 6]$$

$$C_{10} = [6^2]$$

$$C_{11} = [1^2, 5^2]$$

$$C_{12} = [2, 10]$$

$$C_{13} = [1, 11]$$

$$C_{14} = [1, 11]$$

Now, Let $a \in C_{13}$, where $a = (2,3,4,5,11,10,7,12,8,6,9)$,

we find the powers of a :

$$a^1 = (2,3,4,5,11,10,7,12,8,6,9) \in C_{13}$$

$$a^2 = (2,4,11,7,8,9,3,5,10,12,6) \in C_{14}$$

$$a^3 = (2,5,7,6,3,11,12,9,4,10,8) \in C_{13}$$

$$a^4 = (2,11,8,3,10,6,4,7,9,5,12) \in C_{13}$$

$$a^5 = (2,10,9,11,6,5,8,4,12,3,7) \in C_{13}$$

$$a^6 = (2,7,3,12,4,8,5,6,11,9,10) \in C_{14}$$

$$a^7 = (2,12,5,9,7,4,6,10,3,8,11) \in C_{14}$$

$$a^8 = (2,8,10,4,9,12,11,3,6,7,5) \in C_{14}$$

$$a^9 = (2,6,12,10,5,3,9,8,7,11,4) \in C_{13}$$

$$a^{10} = (2,9,6,8,12,7,10,11,5,4,3) \in C_{14}$$

$$a^{11} = e$$

Let $a \in C_{14}$, where

$a = (2,3,4,5,10,6,8,12,11,9,7)$, we find the powers of a :

$$a^1 = (2,3,4,5,10,6,8,12,11,9,7) \in C_{14}$$

$$a^2 = (2,4,10,8,11,7,3,5,6,12,9) \in C_{13}$$

$$a^3 = (2,5,8,9,3,10,12,7,4,6,11) \in C_{14}$$

$$a^4 = (2,10,11,3,6,9,4,8,7,5,12) \in C_{14}$$

$$a^5 = (2,6,7,10,9,5,11,4,12,3,8) \in C_{14}$$

$$a^6 = (2,8,3,12,4,11,5,9,10,7,6) \in C_{13}$$

$$a^7 = (2,12,5,7,8,4,9,6,3,11,10) \in C_{13}$$

$$a^8 = (2,11,6,4,7,12,10,3,9,8,5) \in C_{13}$$

$$a^9 = (2,9,12,6,5,3,7,11,8,10,4) \in C_{14}$$

$$a^{10} = (2,7,9,11,12,8,6,10,5,4,3) \in C_{13}$$

$$a^{11} = e$$

The other conjugate classes can be found by result in S. A. Taban⁽¹¹⁾ as in mentioned in the introduction.

From the powers of the conjugate classes for M_{12} (table 3), we can find The roots of M_{12} as in (table 4):

References

- (1) Frobenius, G., 1903. Über einen Fundamentalsatz der Gruppentheorie. Berl. Sitz, pp.987-991.
- (2) Burnside, W. (1955) "Theory of groups of finite order", Dover publications, second edition,.
- (3) Iiyori and H. Yamaki, N. (1996) "A conjecture of Frobenius", Sugaku Exposition, Vol.9, No.1, Jun.
- (4) Huppert, B. (1967). "Endliche Gruppen". I, Springer, Berlin.
- (5) Hall, P. (1978) "On theorem of Frobenius", Proc. London. Math. Soc. (2) 40(1935/36), 468-480, 481-501.
- (6) Solomon, L. (1969) "The solution of equations in groups", Arch. Math. (Basel) 20, 241-247.
- (7) Lam, T. (1988). "On the number of solution of $x^{p^k} = a$ in a p-group", Illinois J. Math. 32, 575-583.
- (8) Yoshida, T. (1993). " $|\text{Hom}(A, G)|$ ", J. Algebra 156, 125-156.
- (9) Chowla, S. Herstein, I. , and Scott, W. (1952). "The solutions of $x^d=1$ in Symmetric groups", Norske. Vid. Selsk. Forh. (Trondheim) 25, 29-31.

(10) James, G. and Kerber, A. (1984). "The representation theory of the Symmetric group", Addison Wesley Publishing, Cambridge University Press.

(11) Taban. S.A. (2007). "Equation in symmetric groups" Ph. D .Thesis "University of Basra.

(12) Kholil, S.M. (2012). "Solutions of the Frobenius class equation in Alternating groups" Ph. D .Thesis "University Sains Malaysia.

(13) John H., Noam D. Elkies, (2006). and Jeremy L. Martin. "The Mathieu group M_{12} and its pseudogroup The powers of M_{11} Table (1)

the conjugate class	1	2	3	4	5
C_0	C_0	C_0	C_0	C_0	C_0
C_1	C_1	C_0	C_1	C_0	C_1
C_2	C_2	C_0	C_2	C_0	C_2
C_3	C_3	C_2	C_3	C_1	C_4
C_4	C_4	C_2	C_4	C_1	C_3
C_5	C_5	C_6	C_5	C_5	C_5
C_6	C_6	C_5	C_6	C_6	C_6
C_7	C_7	C_7	C_7	C_7	C_0
C_8	C_8	C_8	C_0	C_8	C_8
C_9	C_9	C_8	C_1	C_8	C_9

The powers of M_{11} Table (1)

the conjugate class	6	7	8	9	10	11
C_0	C_0	C_0	C_0	C_0	C_0	C_0
C_1	C_0	C_1	C_0	C_1	C_0	C_1
C_2	C_0	C_2	C_0	C_2	C_0	C_2
C_3	C_2	C_4	C_0	C_3	C_2	C_3
C_4	C_2	C_3	C_0	C_4	C_2	C_4
C_5	C_6	C_6	C_6	C_5	C_6	C_0
C_6	C_5	C_5	C_5	C_6	C_5	C_0
C_7	C_7	C_7	C_7	C_7	C_0	C_7
C_8	C_0	C_8	C_8	C_0	C_8	C_8
C_9	C_0	C_9	C_8	C_1	C_8	C_9

extension M_{13} ". Experimental Mathematics 15, (2): 223-236.

(14) Klymenko, A . (2012). "Multiply transitive permutation groups via the small Mathieu groups" University of Primorska Master Thesis.

(15) James, G.D. (1973) . "The modular characters of Mathieu groups" Journal of algebra 27, 51-111.

The roots of M_{11} Table (2)

the conjugate class	1	2	3	4	5
C_0	C_0	C_0, C_1, C_2	C_0, C_8	C_0, C_1, C_2	C_0, C_7
C_1	C_1	—	C_1, C_9	C_3, C_4	C_1
C_2	C_2	C_3, C_4	C_2	—	C_2
C_3	C_3	—	C_3	—	C_4
C_4	C_4	—	C_4	—	C_3
C_5	C_5	C_6	C_5	C_5	C_5
C_6	C_6	C_5	C_6	C_6	C_6
C_7	C_7	C_7	C_7	C_7	—
C_8	C_8	C_8, C_9	—	C_8, C_9	C_8
C_9	C_9	—	—	—	C_9

The roots of M_{11} Table (2)

the conjugate class	6	7	8	9	10	11
C_0	C_0, C_1, C_2, C_8, C_9	C_0	C_0, C_1, C_2, C_3, C_4	C_0, C_8	C_0, C_1, C_2, C_7	C_0, C_5, C_6
C_1	—	C_1	—	C_1, C_9	—	C_1
C_2	C_3, C_4	C_2	—	C_2	C_3	C_2
C_3	—	C_4	—	C_3	—	C_3
C_4	—	C_3	—	C_4	—	C_4
C_5	C_6	C_6	C_6	C_5	C_6	—
C_6	C_5	C_5	C_5	C_6	C_5	—
C_7	C_7	C_7	C_7	C_7	—	C_7
C_8	—	C_8	C_8, C_9	—	C_8, C_9	C_8
C_9	—	C_9	—	—	—	C_9

The powers of M_{12} Table (3)

The conjugate class	6	7	8	9	10	11
C_0	$[1^{12}]$	C_0	C_0	C_0	C_0	C_0
C_1	$[1^4, 2^4]$	C_0	C_1	C_0	C_1	C_1
C_2	$[2^6]$	C_0	C_2	C_0	C_2	C_2
C_3	$[1^4, 4^2]$	C_1	C_3	C_0	C_3	C_3
C_4	$[2^2, 4^2]$	C_1	C_4	C_0	C_4	C_4
C_5	$[1^2, 2, 8]$	C_3	C_5	C_0	C_5	C_5
C_6	$[4, 8]$	C_4	C_6	C_0	C_6	C_5, C_6
C_7	$[1^3, 3^3]$	C_0	C_7	C_7	C_0	C_7
C_8	$[3^4]$	C_0	C_8	C_8	C_0	C_8
C_9	$[1, 2, 3, 6]$	C_0	C_9	C_7	C_1	C_7
C_{10}	$[6^2]$	C_0	C_{10}	C_8	C_2	C_8
C_{11}	$[1^2, 5^2]$	C_{11}	C_{11}	C_{11}	C_{11}	C_0
C_{12}	$[2, 10]$	C_{11}	C_{12}	C_{11}	C_{12}	C_0
C_{13}	$[1, 11]$	C_{14}	C_{14}	C_{14}	C_{13}	C_{14}
C_{14}	$[1, 11]$	C_{13}	C_{13}	C_{13}	C_{14}	C_0

The powers of M_{12} Table (3)

The conjugate class	2	3	4	5
C_0	$[1^{12}]$	C_0	C_0	C_0
C_1	$[1^4, 2^4]$	C_0	C_1	C_0
C_2	$[2^6]$	C_0	C_2	C_0
C_3	$[1^4, 4^2]$	C_1	C_3	C_0
C_4	$[2^2, 4^2]$	C_1	C_4	C_0
C_5	$[1^2, 2, 8]$	C_0	C_5	C_1
C_6	$[4, 8]$	C_4	C_6	C_1
C_7	$[1^3, 3^3]$	C_7	C_0	C_7
C_8	$[3^4]$	C_8	C_0	C_8
C_9	$[1, 2, 3, 6]$	C_7	C_1	C_7
C_{10}	$[6^2]$	C_8	C_2	C_8
C_{11}	$[1^2, 5^2]$	C_{11}	C_{11}	C_0
C_{12}	$[2, 10]$	C_{11}	C_{12}	C_{11}
C_{13}	$[1, 11]$	C_{14}	C_{13}	C_{13}
C_{14}	$[1, 11]$	C_{13}	C_{14}	C_{14}

From the powers of the conjugate classes

for M_{12} (table 3), we can find The roots

of M_{12} as in (table 4):

conjugate classes		2	3	4	5
C_0	$[1^{12}]$	C_0, C_1, C_2, C_5	C_0, C_7, C_8	C_0, C_1, C_2, C_3, C_4	C_0, C_{11}
C_1	$[1^4, 2^4]$	C_4, C_5	C_1, C_9	C_5, C_6	C_1
C_2	$[2^6]$	—	C_2, C_{10}	—	C_2, C_{12}
C_3	$[1^4, 4^2]$	—	C_3	—	C_3
C_4	$[2^2, 4^2]$	C_6	C_4	—	C_4
C_5	$[1^2, 2, 8]$	—	C_5	—	C_5
C_6	$[4, 8]$	—	C_6	—	C_6
C_7	$[1^3, 3^3]$	C_7, C_9	—	C_7, C_9	C_7
C_8	$[3^4]$	C_8, C_{10}	—	C_8, C_{10}	C_8
C_9	$[1, 2, 3, 6]$	—	—	—	C_9
C_{10}	$[6^2]$	—	—	—	C_{10}
C_{11}	$[1^2, 5^2]$	C_{11}, C_{12}	C_{11}	C_{11}, C_{12}	—
C_{12}	$[2, 10]$	—	C_{12}	—	—
C_{13}	$[1, 11]$	C_{14}	C_{13}	C_{14}	—
C_{14}	$[1, 11]$	C_{13}	C_{14}	C_{13}	C_{13}, C_{14}

Table(4) the roots of M_{12}

conjugate classes		6	7	8	9	10	11
C_0	$[1^{12}]$	$C_0, C_1, C_2, C_7, C_8, C_9, C_{10}$	C_0	C_0, C_1, C_2, C_3, C_4	C_0, C_7, C_8	$C_0, C_1, C_2, C_{11}, C_{12}$	C_0, C_{14}, C_{15}
C_1	$[1^4, 2^4]$	C_3, C_4	C_1	—	C_1, C_9	C_3, C_4	C_1
C_2	$[2^6]$	—	C_2	—	C_2, C_{10}	—	C_2
C_3	$[1^4, 4^2]$	C_5	C_3	C_{11}	C_3	C_5	C_3
C_4	$[2^2, 4^2]$	C_6	C_4	—	C_4	C_6	C_4
C_5	$[1^2, 2, 8]$	—	C_5	—	C_5	—	C_5
C_6	$[4, 8]$	—	C_6	—	C_6	—	C_6
C_7	$[1^3, 3^3]$	—	C_7	C_7, C_8	—	C_7, C_9	C_7
C_8	$[3^4]$	—	C_8	—	—	C_8, C_{10}	C_8

C_9	[1,2,3,6]	—	C_9	—	—	—	C_9
C_{10}	[6 ²]	—	C_{10}	—	—	—	C_{10}
C_{11}	[1 ² , 5 ²]	$C_{11}C_{12}$	C_{11}	C_{11}, C_{12}	C_{11}	—	C_{11}
C_{12}	[2,10]	—	C_{12}	—	C_{12}	—	C_{12}
C_{13}	[1,11]	C_{14}	C_{14}	C_{13}, C_{14}	C_{13}	C_{14}	—
C_{14}	[1,11]	C_{13}	C_{13}	—	C_{14}	C_{13}	—

رنا حسن شمخي

سعيد عبد الأمير تعبان

قسم الرياضيات - كلية العلوم - جامعة البصرة

المستخلص :

في هذا البحث تم ايجاد الحلول للمعادلة $x^n = a$ في زمير الماتثيو M_{11}, M_{12}

حيث ان $n, a \in M_{11} \text{ or } M_{12}$ عدد صحيح موجب.