

Homological Characterization of Free Pro- $\mathbb{C}$ Crossed Modules

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Abstract

Recently several new methods have been introduced into cohomological and combinatorial group theory based on the use of a crossed module. Many of the components of these methods go back to work of Reirdermeister ⁽¹⁾, Peiffer ⁽²⁾ and most notably of J.H.C. Whitehead ⁽³⁾ who introduced crossed modules. The main purpose of this paper is to give a homological characterization of free Pro- $\mathbb{C}$ crossed modules (see theorem 3.3).

Keywords: Free Pro- $\mathbb{C}$ crossed modules; completed group algebra; Pseudocompact module; Homology.

1. Introduction

1.1. Let $\mathbb{C}$ be a non- trivial full subcategory of the category $FGrps$ of finite groups. We will assume that $\mathbb{C}$ is closed under the formation of subgroups, quotients and finite products. Equivalently we assume that the Pro- $\mathbb{C}$ groups (that is the topological groups which are inverse limits of discrete groups from $\mathbb{C}$), form a variety, Pro- $\mathbb{C}$, in the category, $ProFGrps$, of profinite groups. We will further need that $\mathbb{C}$ is also closed under extensions.

Given a profinite space X (i.e. compact, Hausdorff and totally disconnected topological space), we will denote by $F_{\mathbb{C}}(X)$ the free Pro- $\mathbb{C}$ group on X , for a construction of this see Gildenhuys and Lim ⁽⁴⁾.

1.2. Let $\widehat{\mathbb{Z}}_{\mathbb{C}}$ be the free Pro- $\mathbb{C}$ group on one generator. This is also the Pro- $\mathbb{C}$ completion of the ring, $\mathbb{Z}$, of integers. The Pro- $\mathbb{C}$ inherits a ring structure from the integers. The ring $\widehat{\mathbb{Z}}_{\mathbb{C}}$ is a topological ring

which is complete, Hausdorff and which admits a system of open neighborhoods of 0 consisting of two-sided ideals for which the corresponding quotient ring is an artin ring. It is thus a pseudocompact ring (the theory of pseudocompact rings and modules can be found in Brumer's paper ⁽⁵⁾). For a given pseudocompact ring R , the obvious topological R -modules to consider are those which themselves are pseudocompact (i.e., those M which are complete Hausdorff topological modules over R and in which there is a system of open neighborhoods of 0 consisting of submodules N such that M/N is of finite length).

The analogue of the group ring in the context of Pro- $\mathbb{C}$ groups is the concept of group algebra

$$\widehat{\mathbb{Z}}_{\mathbb{C}}[[G]] = \varprojlim_U \widehat{\mathbb{Z}}_{\mathbb{C}}[[G/U]]$$

where G is a Pro- $\mathbb{C}$ group and $U \triangleleft G$ such that $G/U \in \mathbb{C}$. We also mention the fact that the pseudocompact $\widehat{\mathbb{Z}}_{\mathbb{C}}[[G]]$ -modules coincide exactly with those Pro- $\mathbb{C}$ abelian groups which are topological G -modules (see Gildenhuys and Mackay ⁽⁶⁾).

1.3. A Pro- $\mathbb{C}$ crossed module is a continuous homomorphism $\partial: C \rightarrow G$ of Pro- $\mathbb{C}$ groups together with a continuous left action of G on C such that the following conditions are satisfied:

(CM1) for all $g \in G, c \in C$

$$\partial({}^g c) = g\partial(c)g^{-1}, \text{ (i.e., } \partial \text{ is } G\text{-equivariant)}$$

(CM2) for all $c_1, c_2 \in C$

$$\partial({}^{c_2}c_1) = c_2 c_1 c_2^{-1}.$$

We refer to the notation (C, G, ∂) as a Pro- $\mathbb{C}$ crossed module.

A (continuous) morphism $(\mu, \eta): (C, G, \partial) \rightarrow (\hat{C}, \hat{G}, \hat{\partial})$ of Pro- $\mathbb{C}$ crossed modules consists of continuous homomorphisms $\mu: C \rightarrow \hat{C}$ and $\eta: G \rightarrow \hat{G}$ such that:

$$(i) \hat{\partial}\mu = \eta\partial$$

$$(ii) \mu({}^g c) = {}^{\eta(g)}\mu(c), \text{ for all } c \in C \text{ and } g \in G.$$

Examples 1.4.

(i) Suppose G is Pro- $\mathbb{C}$ group and M is a pseudocompact left $\hat{\mathbb{Z}}_{\mathbb{C}}[[G]]$ -module. Let be the trivial map sending everything in M to the identity of G , 1_G , then (M, G, O) is a Pro- $\mathbb{C}$ crossed module.

(ii) Suppose given a continuous homomorphism $\theta: M \rightarrow N$ of pseudocompact left $\hat{\mathbb{Z}}_{\mathbb{C}}[[G]]$ -modules, and form the semidirect product $G \ltimes N$. This is a Pro- $\mathbb{C}$ group which acts continuously on the left of M via the projection from $G \ltimes N$ to G . Define a continuous morphism $\partial: M \rightarrow G \ltimes N$ by $\partial(m) = (1_G, \theta(m))$ for all $m \in M$, then $(M, G \ltimes N, \partial)$ is a Pro- $\mathbb{C}$ crossed module.

We recall the following elementary properties of Pro- $\mathbb{C}$ crossed modules (see Korkes and Porter ⁽⁷⁾).

We suppose given a Pro- $\mathbb{C}$ crossed module $\partial: C \rightarrow G$ with $N = \text{Im}\partial$ and $A = \text{Ker}\partial$.

(i) N is a closed normal subgroup of G .

(ii) A lies in the center of G , hence is an abelian Pro- $\mathbb{C}$ group.

(iii) The closed normal subgroup N of G acts trivially on the left of A , hence A is naturally a pseudocompact left $\hat{\mathbb{Z}}_{\mathbb{C}}[[G/N]]$ -module.

(iv) There are natural pseudocompact left $\hat{\mathbb{Z}}_{\mathbb{C}}[[G/N]]$ -module structures on C^{Ab} and N^{Ab} , the Pro- $\mathbb{C}$ abelianizations of C and N respectively, and thus a Pro- $\mathbb{C}$ crossed module gives one three pseudocompact left $\hat{\mathbb{Z}}_{\mathbb{C}}[[G/N]]$ -modules which are linked by an exact sequence

$$A \rightarrow C^{Ab} \rightarrow N^{Ab} \rightarrow 0.$$

Noting that $\text{Ker}(A \rightarrow C^{Ab}) = A \cap [C, C]$, where $[C, C]$ is the closed commutator subgroup of C . This is zero if the continuous epimorphism from C to N be splits; this happens for instance if N is a free Pro- $\mathbb{C}$ group.

2. Free Pro- $\mathbb{C}$ Crossed Module

Definition 2.1. Suppose C is a Pro- $\mathbb{C}$ G -crossed module and $\{c_\alpha\}_{\alpha \in \Gamma}$ is an indexed set of elements of C , then C is called a free Pro- $\mathbb{C}$ crossed module with basis $\{c_\alpha\}$ if given a Pro- $\mathbb{C}$ G -crossed module $\hat{C}$ with an indexed set $\{\hat{c}_\alpha\}_{\alpha \in \Gamma}$ of elements of $\hat{C}$ and a continuous homomorphism $h: G \rightarrow \hat{G}$ such that $h(\partial(c_\alpha)) = \hat{\partial}(\hat{c}_\alpha)$ for all $\alpha \in \Gamma$, then there is a unique continuous homomorphism $\varphi: C \rightarrow \hat{C}$ such that $\varphi(c_\alpha) = \hat{c}_\alpha$ for all

$\alpha \in \Gamma$, and (φ, h) is a continuous morphism of Pro- $\mathbb{C}$ crossed modules.

We next give the construction of the standard free Pro- $\mathbb{C}$ crossed module (the abstract case is due to J.H.C. Whitehead).

Let G be a Pro- $\mathbb{C}$ group and $\{g_\alpha\}_{\alpha \in \Gamma}$ be an indexed set of elements of G . Let $F_{\mathbb{C}}$ be the free Pro- $\mathbb{C}$ group on $G \times \Gamma = \{(g, \alpha); g \in G, \alpha \in \Gamma\}$.

Let C be the quotient of $F_{\mathbb{C}}$ by the closed normal closure of $\{(x, \alpha)(y, \beta)(x, \alpha)^{-1}(xg_\alpha x^{-1}y, \beta)^{-1}; \alpha, \beta \in \Gamma\}$. It is easy to check that C is a Pro- $\mathbb{C}$ G -crossed module with a continuous action of G induced by the continuous action of G on $F_{\mathbb{C}}$ given by $^g(x, \alpha) = (gx, \alpha)$, and a continuous boundary $\partial: C \rightarrow G$ induced by the continuous homomorphism $\hat{\partial}: F_{\mathbb{C}} \rightarrow G$ given by $\hat{\partial}(x, \alpha) = xg_\alpha x^{-1}$. Clearly, the image of ∂ is the closed normal closure of $\{g_\alpha\}_{\alpha \in \Gamma}$ in G . Let $p: F_{\mathbb{C}} \rightarrow C$ be the natural projection, and let $c_\alpha = p(1, \alpha)$ for all $\alpha \in \Gamma$. Observe that if $r \in F_{\mathbb{C}}$, then r is of the form $r = \prod_{i=1}^n (\alpha_i, x_i)^{\epsilon_i}$ where $\epsilon_i = \pm 1$; hence $p(r) = {}^{x_1}c_{\alpha_1}^{\epsilon_1} {}^{x_2}c_{\alpha_2}^{\epsilon_2} \dots {}^{x_n}c_{\alpha_n}^{\epsilon_n}$. In other words, $\{c_\alpha\}$ generates C (topologically) in the operator sense. We call C the standard free Pro- $\mathbb{C}$ G -crossed module with boundaries $\{g_\alpha\}$, and $\{c_\alpha\}$ the standard basis.

Let us now state three elementary results on free Pro- $\mathbb{C}$ crossed modules (the proofs are straightforward, see ⁽⁷⁾).

Proposition 2.2. Suppose C is a Pro- $\mathbb{C}$ G -crossed module and $\{c_\alpha\}_{\alpha \in \Gamma}$ is an indexed set of elements of C , then the following statements are equivalent:

(i) C is a free Pro- $\mathbb{C}$ G -crossed module with basis $\{c_\alpha\}$,

(ii) For every Pro- $\mathbb{C}$ G -crossed module $\hat{C}$ and an indexed subset of elements $\{\hat{c}_\alpha\}$ such that $\partial(\hat{c}_\alpha) = \hat{c}_\alpha$ for all $\alpha \in \Gamma$, there is a unique continuous G -homomorphism $\varphi: C \rightarrow \hat{C}$ such that $\varphi(c_\alpha) = \hat{c}_\alpha$ for all $\alpha \in \Gamma$,

(iii) C is G -isomorphic to the standard free Pro- $\mathbb{C}$ G -crossed module with boundaries $\{\partial(c_\alpha)\}$ in such a way that $\{c_\alpha\}$ corresponds to the standard basis.

Corollary 2.3. Suppose C is a free Pro- $\mathbb{C}$ G -crossed module with basis $\{c_\alpha\}$, then $\{c_\alpha\}$ generates C (topologically) in the operator sense.

Proposition 2.4. Suppose C is a free Pro- $\mathbb{C}$ G -crossed module with basis $\{c_\alpha\}$, then C^{Ab} is a free pseudocompact left $\mathbb{Z}_{\mathbb{C}}[[G/N]]$ -module with basis $\{\bar{c}_\alpha\}_{\alpha \in \Gamma}$, where $\bar{c}_\alpha$ is the image of c_α under the continuous natural projection $\rho: C \rightarrow C^{Ab}$.

3. Homological Characterization of Free Pro- $\mathbb{C}$ Crossed Modules

3.1. Suppose G is a Pro- $\mathbb{C}$ group and $\{g_\alpha\}_{\alpha \in \Gamma}$ is an indexed set of elements in G . Let $F_{\mathbb{C}}$ be the free Pro- $\mathbb{C}$ group on $G \times \Gamma$ and consider the continuous morphism $\hat{\partial}: F_{\mathbb{C}} \rightarrow G$ given by $\hat{\partial}(x, \alpha) = xg_\alpha x^{-1}$. The elements $I = \text{Ker } \hat{\partial}$ are called the identities among the boundaries $\{g_\alpha\}$. A typical element of $F_{\mathbb{C}}$ is of the form $\prod_{i=1}^n (x_i, \alpha_i)^{\epsilon_i}$ with $\epsilon_i = \pm 1$; moreover, $\prod_{i=1}^n (x_i, \alpha_i)^{\epsilon_i}$ is an identity if, and only if, $\prod_{i=1}^n x_i g_{\alpha_i}^{\epsilon_i} x_i^{-1} = 1$. The standard free Pro- $\mathbb{C}$ G -crossed module C with boundaries $\{g_\alpha\}$ is the quotient of $F_{\mathbb{C}}$ by the closed normal closure of $E = \{(x, \alpha)(y, \beta)(x, \alpha)^{-1}(xg_\alpha x^{-1}y, \beta)^{-1}; x, y \in G \text{ and } \alpha, \beta \in \Gamma\}$. Let $p: F_{\mathbb{C}} \rightarrow C$ be the continuous natural projection then the continuous boundary of C is defined by

$\partial \circ p = \hat{\partial}$. Hence $\text{Ker } p = \text{Ker } \hat{\partial}$. Observe that $(xg_\alpha x^{-1})(yg_\beta y^{-1})(xg_\alpha^{-1}x^{-1})$

$$\left((xg_\alpha x^{-1}y)g_\beta^{-1}(xg_\alpha x^{-1}y)^{-1} \right) = 1.$$

The elements of $P = \text{Ker } p$ are called the Peiffer identities among the boundaries $\{g_\alpha\}$. Observe that $C = F_{\mathbb{C}}/P$,

where P is a closed normal G -invariant of $F_{\mathbb{C}}$ (see ⁽⁷⁾), and $\text{Ker } \partial = I/P$. So that I/P is a pseudocompact left $\widehat{\mathbb{Z}}_{\mathbb{C}}[[G/N]]$ -module with a continuous left action of G/N induced by the continuous left action of G on C .

Lemma 3.2. ⁽⁸⁾ Suppose G is a Pro- $\mathbb{C}$ group and $\{g_\alpha\}_{\alpha \in \Gamma}$ is an indexed set of elements in G . Suppose $F_{\mathbb{C}}$ is the free Pro- $\mathbb{C}$ group on $G \times \Gamma$ and I and P are the identities and Peiffer identities in $F_{\mathbb{C}}$. Then $P \cap [F_{\mathbb{C}}, F_{\mathbb{C}}] = [F_{\mathbb{C}}, I]$.

In the following theorem, we give a homological characterization of the free Pro- $\mathbb{C}$ crossed module.

Theorem 3.3. Suppose C is a Pro- $\mathbb{C}$ G -crossed module and $\{c_\alpha\}_{\alpha \in \Gamma}$ is an indexed set of elements of C , the C is a free Pro- $\mathbb{C}$ crossed module if, and only if,

- (i) C^{Ab} is free pseudocompact left $\widehat{\mathbb{Z}}_{\mathbb{C}}[[G/N]]$ -module with basis $\{\bar{c}_\alpha\}$,
- (ii) N is the closed normal closure of $\{\partial(c_\alpha)\}$ in G ,
- (iii) $\partial_*: H_2(C) \rightarrow H_2(N)$ is trivial.

Proof. Suppose C is a free Pro- $\mathbb{C}$ G -crossed module with basis $\{c_\alpha\}$. Statement (i) is true according to proposition (2.4), while (ii) follows from corollary (2.3), so we need only to prove (iii). From the elementary properties (i) and (ii) of Pro- $\mathbb{C}$ crossed modules, we deduce that the central Pro- $\mathbb{C}$ extension $A \twoheadrightarrow C \twoheadrightarrow N$ which

determine a 5-term homology exact sequence ⁽¹⁰⁾.

$$\begin{aligned} H_2(C) &\xrightarrow{\partial_*} H_2(N) \rightarrow A \rightarrow H_1(C) \\ &\xrightarrow{\partial_*} H_1(N) \rightarrow 0. \end{aligned}$$

We remark that the 5-term stalling ⁽⁹⁾ and Stambach ⁽¹¹⁾ sequence also hold for Pro- $\mathbb{C}$ groups ⁽⁷⁾. Note that we have a pro- $\mathbb{C}$ presentation for N and C ,

$$\begin{aligned} 1 \rightarrow I \rightarrow F_{\mathbb{C}} &\xrightarrow{\hat{\partial}} N \rightarrow 1 \\ 1 \rightarrow P \rightarrow F_{\mathbb{C}} &\xrightarrow{p} C \rightarrow 1. \end{aligned}$$

According to Eckman, Hilton and Stambach ⁽¹⁰⁾, the 5-term homology exact sequence is equivalent to the sequence

$$\begin{aligned} \frac{P \cap [F_{\mathbb{C}}, F_{\mathbb{C}}]}{[F_{\mathbb{C}}, P]} &\xrightarrow{\partial_*} \frac{I \cap [F_{\mathbb{C}}, F_{\mathbb{C}}]}{[F_{\mathbb{C}}, I]} \rightarrow \frac{I}{P} \rightarrow C^{Ab} \\ &\rightarrow N^{Ab} \rightarrow 0 \end{aligned}$$

which uses Hopf's formula for the second homology group. The first two continuous homomorphisms are induced by inclusion. By lemma (3.2) $P \cap [F_{\mathbb{C}}, F_{\mathbb{C}}] = [F_{\mathbb{C}}, I]$. This implies that the first continuous homomorphism is trivial, and hence $\partial_*: H_2(C) \rightarrow H_2(N)$ is trivial.

Conversely, suppose C and $\{c_\alpha\}$ satisfy (i)-(iii) above. Let $\hat{C}$ be the standard free Pro- $\mathbb{C}$ G -crossed module with boundaries $\{\partial(c_\alpha)\}$ and standard basis $\{\hat{c}_\alpha\}$. Observe that $\hat{\partial}(\hat{c}_\alpha) = \partial(c_\alpha)$ for all $\alpha \in \Gamma$ by construction. By (ii), $\hat{N} = N$. By proposition (2.2-ii), there is a continuous G -homomorphism $\varphi: \hat{C} \rightarrow C$ such that $\varphi(c_\alpha) = \hat{c}_\alpha$ for all $\alpha \in \Gamma$. Since φ is a continuous G -homomorphism and $\hat{\partial} = \partial \circ \varphi$, hence $\varphi(\hat{A}) \subset A$. Thus φ induces a continuous homomorphism $\varphi_*: \hat{A} \rightarrow A$.

The 5-term homology sequence is natural. Hence, the following diagram commute:

$$\begin{array}{ccccccc} 0 & \rightarrow & H_2(N) & \rightarrow & \hat{A} & \rightarrow & \hat{C}^{Ab} \xrightarrow{\bar{\delta}} N^{Ab} \rightarrow 0 \\ & & \parallel & & \varphi \downarrow & & \bar{\varphi} \downarrow & & \parallel & & (\hat{N} = N) \\ 0 & \rightarrow & H_2(N) & \rightarrow & A & \rightarrow & C^{Ab} \xrightarrow{\bar{\delta}} N^{Ab} \rightarrow 0 \end{array}$$

Moreover, the connecting homomorphisms $\bar{\delta}$ and $\bar{\delta}$ are continuous monomorphisms by (iii). Hence the top and bottom rows are exact.

By (i), C^{Ab} is free pseudocompact left $\hat{\mathbb{Z}}_{\mathbb{C}}[[G/N]]$ -module with basis $\{\bar{c}_{\alpha}\}$, hence

$\bar{\varphi}(\bar{c}_{\alpha}) = \bar{c}_{\alpha}$ for all $\alpha \in \Gamma$ implies that $\bar{\varphi}$ is a continuous isomorphism. Therefore, $\varphi \circ$ is a continuous isomorphism by the 5-lemma. Observe that the following diagram commutes:

$$\begin{array}{ccccccc} 0 & \rightarrow & \hat{A} & \rightarrow & \hat{C} & \rightarrow & N \rightarrow 1 \\ & & \varphi \downarrow & & \varphi \downarrow & & \parallel & & & & (\text{where } \hat{N} = N) \\ 0 & \rightarrow & A & \rightarrow & C & \rightarrow & N \rightarrow 1 \end{array}$$

Hence, φ is a continuous isomorphism by the short 5-lemma. It follows that $\hat{C}$ is G -isomorphic to C via φ . Since $\varphi(\hat{c}_{\alpha}) = c_{\alpha}$ for all $\alpha \in \Gamma$, C is a free Pro- $\mathbb{C}$ G -crossed module with basis $\{c_{\alpha}\}$.

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الميزات الهومولوجية للموديولات المتصالبة $\mathbb{C}$ - الاسقاطية الحرة

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الخلاصة:

استحدثت مؤخراً طرائق عديدة في نظرية الزمر الهومولوجية المتمتمة ونظرية الزمر التوافقية، ارتكزت حول استخدام مفهوم الموديول المتصالب. ان معظم ركائز هذه الطرائق تعود الى Peiffer، Reidermeister وبصورة اساسية الى J. H. C. Whitehead الذي استحدث مفهوم الموديولات المتصالبة. الهدف الاساسي من هذا البحث هو اعطاء الميزات الهومولوجية للموديولات المتصالبة $\mathbb{C}$ - الاسقاطية الحرة .

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