

The Voronovskaya Theorem for q - Analogue of Szasz-Mirakjan Operators with Two Variables

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Abstract

In this paper, we introduce and study new two operators of q –Szász–Mirakyan in two dimensional space which are denote by $(R_n)_\lambda(f(t, s), q_n, \tilde{q}_n; x, y), \lambda = 1, 2$ respectively, where $f \in C_{\alpha, \beta}([0, \infty) \times [0, \infty))$, and $x, y \in [0, \infty)$ and $q_n, \tilde{q}_n \in (0, 1); q_n, \tilde{q}_n \rightarrow 0$ as $n \rightarrow \infty$. First, we prove that these operators are converging to the function being approximated. Then, we define the m – th and p –th order q –moments. Finally, we prove Voronovskaya- type asymptotic formulas for the operators $(R_n)_\lambda(f(t, u), q_n, \tilde{q}_n; x, y), \lambda = 1, 2$.

Keywords: q –Szász–Mirakyan; q –integers; linear positive operators; sequence of positive numbers; m -th order q –moments; Voronovskaya- type asymptotic formulas.

1. INTRODUCTION:

Suppose that $C[0, \infty)$ denotes the space of all continuous real-valued functions on the interval $[0, \infty)$ the space $C_\alpha[0, \infty)$ of the space $C[0, \infty)$ is defined as:

$$C_\alpha[0, \infty) = \{f \in C[0, \infty): f(t) = O(e^\alpha), \text{ for some } \alpha > 0\}.$$

The space $C_\alpha[0, \infty)$ in normed by the norm:

$$\|f\|_{C_\alpha} = \sup_{t \in [0, \infty)} |f(t)| e^{-\alpha t}, \alpha > 0$$

On the area $([0, \infty) \times [0, \infty))$, $C_{\alpha, \beta}([0, \infty) \times [0, \infty))$ denotes the space of all continuous function such that $|f(t, u)| \leq C e^{\alpha_1(t+u)}$, for some $C > 0$, and $\alpha_1 = \max\{\alpha, \beta\}$ where $\alpha, \beta > 0$. $x, y \in [0, \infty)$ are

two independent variables, and $(q_n - 1) = O([n]^{-2})$ and $(\tilde{q}_n - 1) = O([n]^{-2})$.

The space $C_{\alpha, \beta}$ is normed by the norm:

$$\|f\|_{C_{\alpha, \beta}} = \sup_{(t, u) \in [0, \infty) \times [0, \infty)} |f(t, u)| e^{-\alpha_1(t+u)}, \alpha_1 = \max\{\alpha, \beta\}.$$

For $f \in C_\alpha[0, \infty)$, L.Rempulsa and M.Skorupka (1997) [6], defined and studied the following Szász–Mirakyan type operators:

$$(L_n)_1(f(t); x) = \sum_{k=0}^{\infty} a_{n,k}(x) f\left(\frac{2k}{n}\right)$$

$$(L_n)_2(f(t); x) = \frac{n}{2} \sum_{k=0}^{\infty} a_{n,k}(x) \int_{I_{n,k}} f(t) dt$$

$$(L_n)_3(f(t); x) = \frac{f(0)}{(1 + \sinh(nx))} + \sum_{k=0}^{\infty} b_{n,k}(x) f\left(\frac{2k+1}{n}\right)$$

$$(L_n)_4(f(t); x) = \frac{f(0)}{(1 + \sinh(nx))} + \frac{n}{2} \sum_{k=0}^{\infty} b_{n,k}(x) \int_{I_{n,k}^*} f(t) dt$$

where $a_{n,k}(x) = \frac{1}{\cosh(nx)} \frac{(nx)^{2k}}{(2k)!}$, $b_{n,k}(x) = \frac{1}{(1+\sinh(nx))} \frac{(nx)^{2k+1}}{(2k+1)!}$, $x \in [0, \infty)$, $I_{n,k} = \left[\frac{2k}{n}, \frac{2k+2}{n}\right]$, $I_{n,k}^* = \left[\frac{2k+1}{n}, \frac{2k+3}{n}\right]$, $n \in N$ and $k \in N^0$

L.Rempulsa and M.Skorupka (1996) [8], (1998) [9], studied the above linear positive operators in two variables. B.Firlej, M.Lesniewicz and L.Rempulsa in (1999) [4], generalized and studied these operators for a function in two variables. Also, Rempulsa and Skorupka in the same year [7], introduced four theorems on the convergence of first derivatives of the operators $(L_n)_\lambda$, $\lambda = 1, 2, 3, 4$, for functions of one and two variables. Abdul-Razaq (2013) [12], study the Voronovskaya theorem for q – Analogue of Szász–Mirakyan type operators in one variable which are defined by:

$$R_{n,q_n}(f; x) = \sum_{k=0}^{\infty} a_{n,k}(q_n, x) f\left(\frac{[2k]_{q_n}}{[n]_{q_n}}\right),$$

$$\tilde{R}_{n,q_n}(f; x)$$

$$= \frac{[n]_{q_n}}{[2]_{q_n}} \sum_{k=0}^{\infty} a_{n,k}(q_n, x) \int_{I_{n,k}} f(t) d_{q_n} t,$$

$$a_{n,k}(q_n, x) = \frac{1}{\cosh_{q_n}([n]_{q_n} x)} \frac{[n]_{q_n}^{2k} x^{2k}}{[2k]_{q_n}!}$$

$$I_{n,k} = \left[\frac{[2k]_{q_n}}{[n]_{q_n}}, \frac{[2k+2]_{q_n}}{[n]_{q_n}}\right]$$

The approximation of function by using linear positive operators introduced via q – Calculus is currently under intensive research. Aral, A. (2008) [1], proposed and studies a generalization of Szász–Mirakyan operators, (1950)[13], involving on the q – integers. Aral, A. and Gupta (2006) [2], Mahmudove N. (2010) [10], studies a modification for q – Szász–Mirakyan operators.

We mention some basic definitions and notations used in q -calculus, details can be found in Ernst, T (2000) [5], Victor Kac Pokman Cheung (2002) [14].

$$[n]_q := 1 + q + \dots + q^{n-1} = \frac{1 - q^n}{1 - q}, \quad (n = 1, 2, \dots), [0]_q = 0$$

$$[n]_q! := [1]_q [2]_q \dots [n]_q \text{ for } n \in N, \text{ and } [0]! = 1$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q! [n-k]_q!}$$

$$(t-x)_q^n = (t-x)_q (t-qx)_q (t-q^2x)_q \dots (t-q^{n-1}x)_q$$

The q -derivative of a function $f(x)$, denoted by $D_q f$, is defined by:

$$(D_q f)(x) := \frac{f(qx) - f(x)}{(q-1)x}, x \neq 0, \text{ and the higher}$$

$$q\text{-derivatives as } D_q^0 f := f,$$

$$D_q^n f := D_q(D_q^{n-1} f), (n = 1, 2, \dots).$$

The formula for the q -derivatives of a product and quotient are

$$\begin{aligned}
& D_q(f(x)g(x)) \\
&= f(qx)D_q(g(x)) \\
&+ g(x)D_q(f(x)) \quad (1.1)
\end{aligned}$$

$$\begin{aligned}
& D_q\left(\frac{f(x)}{g(x)}\right) \\
&= \frac{g(qx)D_q f(x) - f(qx)D_q g(x)}{g(x)g(qx)} \quad (1.2)
\end{aligned}$$

The q -exponential function define as: $e_q(x) = \sum_{k=0}^{\infty} \frac{x^k}{[k]_q!}$ and the q -derivative of exponential function is defined by $D_q e_q^{nx} = [n]e_q^{nx}$. The q - analogues of hyperbolic functions (Ernst, T (2000) [5],

$$\cosh_q([n]x) = \frac{e_q^{nx} + e_q^{-nx}}{2} = \sum_{k=0}^{\infty} \frac{[n]^{2k} x^{2k}}{[2k]_q!};$$

$$\begin{aligned}
\sinh_q([n]x) &= \frac{e_q^{nx} - e_q^{-nx}}{2} \\
&= \sum_{k=0}^{\infty} \frac{[n]^{2k+1} x^{2k+1}}{[2k+1]_q!};
\end{aligned}$$

$$\tanh_q([n]x) = \frac{\sinh_q([n]x)}{\cosh_q([n]x)} = \frac{e_q^{nx} - e_q^{-nx}}{e_q^{nx} + e_q^{-nx}}.$$

In our work, we define and study the following q - Szász–Mirakyan operators with two variables:

$$(R_n)_1(f(t, u), q_n, \tilde{q}_n; x, y) =$$

$$\begin{aligned}
& \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a_{n,k}(q_n, x) a_{n,j}(\tilde{q}_n, y) \\
& \times f\left(\frac{[2k]_{q_n}}{[n]_{q_n}}, \frac{[2j]_{\tilde{q}_n}}{[n]_{\tilde{q}_n}}\right) \quad (1.3)
\end{aligned}$$

$$(R_n)_2(f(t, u), q_n, \tilde{q}_n; x, y) = \frac{[n]_{q_n}}{[2]_{q_n}} \frac{[n]_{\tilde{q}_n}}{[2]_{\tilde{q}_n}} \times$$

$$\begin{aligned}
& \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a_{n,k}(q_n, x) a_{n,j}(\tilde{q}_n, y) \\
& \times \int_{I_{n,k}} \int_{I_{n,j}} f(t, u) dt du \quad (1.4)
\end{aligned}$$

$$\text{where } a_{n,k}(q_n, x) = \frac{1}{\cosh_{q_n}([n]x)} \frac{[n]_{q_n}^{2k} x^{2k}}{[2k]_{q_n}!},$$

$$a_{n,j}(\tilde{q}_n, y) = \frac{1}{\cosh_{\tilde{q}_n}([n]y)} \frac{[n]_{\tilde{q}_n}^{2j} y^{2j}}{[2j]_{\tilde{q}_n}!},$$

$$\text{and } I_{n,k} = \left[\frac{[2k]_{q_n}}{[n]_{q_n}}, \frac{[2k+2]_{q_n}}{[n]_{q_n}} \right],$$

$$I_{n,j} = \left[\frac{[2j]_{\tilde{q}_n}}{[n]_{\tilde{q}_n}}, \frac{[2j+2]_{\tilde{q}_n}}{[n]_{\tilde{q}_n}} \right].$$

In this paper we study some properties of the operators $(R_n)_i(f(t, u), q_n, \tilde{q}_n; x, y), i = 1, 2$ i.e. the convergence theorem, m -th and p -th order q -moments then we study Voronovskaya type asymptotic formula for these operators.

2. Preliminary Results

For the weight functions $a_{n,k}(q_n, x)$ of q_n -Szász-Mirakjan Operators we give the following lemma:

Lemma 2.1: Abdul-Razaq (2013) [12]

For $x \in (0, \infty)$, we have:

$$(1) \sum_{k=0}^{\infty} a_{n,k}(q_n, x) = 1;$$

$$\begin{aligned}
(2) \sum_{k=0}^{\infty} a_{n,k}(q_n, x) [2k]_{q_n} \\
= [n]_{q_n} x \tanh_{q_n}([n]x);
\end{aligned}$$

$$(3) \sum_{k=0}^{\infty} a_{n,k}(q_n, x) [2k]_{q_n}^2 \\ = [n]_{q_n} x \tanh_{q_n}([n]x) \\ + q_n [n]_{q_n}^2 x^2;$$

$$(4) x D_{q_n} a_{n,k}(q_n, x) = [2k]_{q_n} a_{n,k}(q_n, x) - \\ [n]_{q_n} x \tanh_{q_n}([n]x) a_{n,k}(q_n, q_n x); \quad (2.1)$$

Now, we give the following lemma, which gives us the properties of q_n –analogue integration

Lemma 2.2: Abdul-Razaq (2013) [12]

$$(1) \int_{\frac{[2k]_{q_n}}{[n]_{q_n}}}^{\frac{[2k+2]_{q_n}}{[n]_{q_n}}} d_{q_n} t = \frac{[2]_{q_n}}{[n]_{q_n}} ((q_n - 1)[2k]_{q_n} \\ + 1);$$

$$(2) \int_{\frac{[2k]_{q_n}}{[n]_{q_n}}}^{\frac{[2k+2]_{q_n}}{[n]_{q_n}}} t d_{q_n} t = \frac{1}{[n]_{q_n}^2 [2]_{q_n}} ([2]_{q_n}^2 \\ + 2[2]_{q_n} q_n^2 [2k]_{q_n} \\ + (q_n^4 - 1)[2k]_{q_n}^2);$$

$$(3) \int_{\frac{[2k]_{q_n}}{[n]_{q_n}}}^{\frac{[2k+2]_{q_n}}{[n]_{q_n}}} t^2 d_{q_n} t \\ = ([2]_{q_n}^3 + 3[2]_{q_n}^2 q_n^2 [2k]_{q_n} \\ + 3[2]_{q_n} q_n^4 [2k]_{q_n}^2 + (q_n^6 - 1)[2k]_{q_n}^3).$$

Lemma 2.3: Mohammad and Abdul-Razaq (2013) [3]

For $m \in N$, the m -th order moment for the operators $(R_n)_\lambda, \lambda = 1, 2$ are defined respectively as:

$$(1) (T_{n,0})_1(x) = 1;$$

$$(T_{n,1})_1(x) = x \tanh_{q_n}([n]x) - x;$$

$$(T_{n,2})_1(x) = \frac{x}{[n]_{q_n}} \tanh_{q_n}([n]x) \\ - [2]_{q_n} x^2 \tanh_{q_n}([n]x) \\ + 2q_n x^2;$$

$$[n]_{q_n} (T_{n,m+1})_1(x) = x D_{q_n} (T_{n,m})_1(x) \\ - [n]_{q_n} q_n^m x (T_{n,m})_1(x)$$

$$+ [n]_{q_n} x \tanh_{q_n}([n]x) (T_{n,m})_1(q_n x) \\ + ([m]x \\ - [n]_{q_n} x^2 \tanh_{q_n}([n]x) (1 \\ - q_n^m)) (T_{n,m-1})_1(q_n x). \quad (2.2)$$

$$(2) (T_{n,0})_2(x) = (q_n - 1)[n]_{q_n} x \tanh_{q_n}([n]_{q_n} x) \\ + 1,$$

$$(T_{n,1})_2(x) \\ = \frac{1}{[n]_{q_n}} + x \left(\frac{([4]_{q_n} - 2)}{[2]_{q_n}} \tanh_{q_n}([n]_{q_n} x) - 1 \right) \\ + (q_n - 1)[n]_{q_n} x^2 \left(\frac{(q_n^2 + 1)q_n}{[2]_{q_n}} \right. \\ \left. - \tanh_{q_n}([n]_{q_n} x) \right),$$

$$(T_{n,2})_2(x) = \frac{1}{[n]_{q_n}} x \times \\ (([4]_{q_n} + q_n - 2) \tanh_{q_n}([n]_{q_n} x) - [2]_{q_n})$$

$$\begin{aligned}
& +x^2 \left(\frac{3q_n^5}{[3]_{q_n}} + q_n^5 - q_n(q_n - 1)(2q_n - 1) \right. \\
& \quad \left. - ([4]_{q_n} - 2) \tanh_{q_n}([n]_{q_n}x) \right) \\
& \quad + q_n(q_n - 1)[n]_{q_n}x^3 \left((q_n^2([3]_{q_n} - 2q_n) \right. \\
& \quad \left. + 1) \tanh_{q_n}([n]x) + ([3]_{q_n} - q_n) \right) + \frac{[2]_{q_n}^2}{[3]_{q_n}[n]_{q_n}^2}
\end{aligned}$$

$$\begin{aligned}
& \frac{[n]_{q_n}[m+2]_{q_n}}{q_n^2[m+1]_{q_n}} (T_{n,m+1})_2(x) \\
& = x D_{q_n} (T_{n,m})_2(x) \\
& \quad - \frac{[n]_{q_n}}{q_n^2} \left(q_n^{m+1}x - \frac{[2]_{q_n}}{[n]_{q_n}} \right) (T_{n,m})_2(x) \\
& \quad + [n]_{q_n}x \tanh_{q_n}([n]_{q_n}x) (T_{n,m})_2(q_nx) \\
& \quad + x (T_{n,m-1})_2(q_nx) ([m]_{q_n} - (1 - q_n^m)[n]_{q_n}x^2 \tanh_{q_n}([n]_{q_n}x)) \\
& \quad + \frac{[n]^2}{[2]_{q_n}[m+1]_{q_n}} \left(q_n^{m+1}x - \frac{1}{q_n^2} \left(q_n^{m+1}x - \frac{[2]_{q_n}}{[n]_{q_n}} \right) \right) (T_{n,m+1})_1(x) \\
& \quad - \frac{[n]_{q_n}^2}{[2]_{q_n}[m+1]_{q_n}} \left(\frac{1}{q_n^2} - 1 \right) (T_{n,m+2})_1(x) \tag{2.3}
\end{aligned}$$

In the next our theorem we prove the operators $(R_n)_\lambda(f(t, u), q_n, \tilde{q}_n; x, y), \lambda = 1, 2$ converges to the function $f(x, y)$ as $n \rightarrow \infty$.

THEOREM 2.1:

Let $n \in N$ for all $x, y \in [0, \infty)$, if the following conditions hold:

$$\begin{aligned}
1. \quad & \lim_{n \rightarrow \infty} \|(R_n)_\lambda(1, q_n, \tilde{q}_n; x, y) - 1\|_{C_{\alpha, \beta}} = 0;
\end{aligned}$$

$$\begin{aligned}
2. \quad & \lim_{n \rightarrow \infty} \|(R_n)_\lambda(t, q_n, \tilde{q}_n; x, y) - x\|_{C_{\alpha, \beta}} = 0;
\end{aligned}$$

$$\begin{aligned}
3. \quad & \lim_{n \rightarrow \infty} \|(R_n)_\lambda(u, q_n, \tilde{q}_n; x, y) - y\|_{C_{\alpha, \beta}} = 0;
\end{aligned}$$

$$\begin{aligned}
4. \quad & \lim_{n \rightarrow \infty} \|(R_n)_\lambda(t^2 + u^2, q_n, \tilde{q}_n; x, y) - (x^2 + y^2)\|_{C_{\alpha, \beta}} = 0,
\end{aligned}$$

then,

$$\begin{aligned}
& \|(R_n)_\lambda(f(t, u), q_n, \tilde{q}_n; x, y) - f(x, y)\|_{C_{\alpha, \beta}} = 0 \text{ as } n \rightarrow \infty, \lambda = 1, 2, 3, 4.
\end{aligned}$$

PROOF:

To prove this theorem we need the properties of the weight functions $a_{n,k}(q_n, x)$ of q_n -Szász-Mirakjan in lemma 2.1 and lemma 2.2:

for $\lambda = 1$

$$\begin{aligned}
(1) \quad & \lim_{n \rightarrow \infty} \|(R_n)_1(1, q_n, \tilde{q}_n; x, y) - 1\|_{C_{\alpha, \beta}} \\
& = \lim_{n \rightarrow \infty} \left\| \sum_{k=0}^{\infty} a_{n,k}(q_n, x) \sum_{j=0}^{\infty} a_{n,j}(\tilde{q}_n, y) - 1 \right\|_{C_{\alpha, \beta}} = 0.
\end{aligned}$$

$$(2) \quad \lim_{n \rightarrow \infty} \|(R_n)_1(t, q_n, \tilde{q}_n; x, y) - x\|_{C_{\alpha, \beta}}$$

$$\begin{aligned}
& = \lim_{n \rightarrow \infty} \left\| \sum_{k=0}^{\infty} a_{n,k}(q_n, x) \left(\frac{[2k]_{q_n}}{[n]_{q_n}} \right) \sum_{j=0}^{\infty} a_{n,j}(\tilde{q}_n, y) - x \right\|_{C_{\alpha, \beta}} \\
& = \lim_{n \rightarrow \infty} \|x \tanh_{q_n}([n]x) - x\|_{C_{\alpha, \beta}} = 0.
\end{aligned}$$

$$(3) \lim_{n \rightarrow \infty} \|(R_n)_1(u, q_n, \tilde{q}_n; x, y) - y\|_{C_{\alpha, \beta}}$$

$$= \lim_{n \rightarrow \infty} \left\| \sum_{k=0}^{\infty} a_{n,k}(q_n, x) \sum_{j=0}^{\infty} a_{n,j}(\tilde{q}_n, y) \left(\frac{[2j]_{\tilde{q}_n}}{[n]_{\tilde{q}_n}} \right) - y \right\|_{C_{\alpha, \beta}}$$

$$= \lim_{n \rightarrow \infty} \|y \tanh_{\tilde{q}_n}([n]y) - y\|_{C_{\alpha, \beta}} = 0.$$

$$(4) \lim_{n \rightarrow \infty} \|(R_n)_1(t^2 + u^2, q_n, \tilde{q}_n; x, y) - (x^2 + y^2)\|_{C_{\alpha, \beta}}$$

$$= \lim_{n \rightarrow \infty} \left\| \left(\sum_{k=0}^{\infty} a_{n,k}(q_n, x) \left(\frac{[2k]_{q_n}}{[n]_{q_n}} \right)^2 \sum_{j=0}^{\infty} a_{n,j}(\tilde{q}_n, y) + \sum_{k=0}^{\infty} a_{n,k}(q_n, x) \sum_{j=0}^{\infty} a_{n,j}(\tilde{q}_n, y) \left(\frac{[2j]_{\tilde{q}_n}}{[n]_{\tilde{q}_n}} \right)^2 \right) - (x^2 + y^2) \right\|_{C_{\alpha, \beta}}$$

$$= \lim_{n \rightarrow \infty} \left\| \left(\frac{1}{[n]_{q_n}} x \tanh_q([n]x) + q_n x^2 + \frac{1}{[n]_{\tilde{q}_n}} y \tanh_{\tilde{q}}([n]y) + \tilde{q}_n y^2 \right) - (x^2 + y^2) \right\|_{C_{\alpha, \beta}} = 0.$$

Now, for $\lambda = 2$, we have:

$$(1) \lim_{n \rightarrow \infty} \|(R_n)_2(1, q_n, \tilde{q}_n; x, y) - 1\|_{C_{\alpha, \beta}}$$

$$= \lim_{n \rightarrow \infty} \left\| \frac{[n]_{q_n}}{[2]_{q_n}} \sum_{k=0}^{\infty} a_{n,k}(q_n, x) \int_{I_{n,k}} d_{q_n} t \frac{[n]_{\tilde{q}_n}}{[2]_{\tilde{q}_n}} \times \sum_{j=0}^{\infty} a_{n,j}(\tilde{q}_n, y) \int_{I_{n,j}} d_{\tilde{q}_n} u - 1 \right\|_{C_{\alpha, \beta}} = 0$$

$$(2) \lim_{n \rightarrow \infty} \|(R_n)_2(t, q_n, \tilde{q}_n; x, y) - x\|_{C_{\alpha, \beta}}$$

$$= \lim_{n \rightarrow \infty} \left\| \frac{[n]_{q_n}}{[2]_{q_n}} \sum_{k=0}^{\infty} a_{n,k}(q_n, x) \int_{I_{n,k}} t d_{q_n} t \frac{[n]_{\tilde{q}_n}}{[2]_{\tilde{q}_n}} \times \sum_{j=0}^{\infty} a_{n,j}(\tilde{q}_n, y) \int_{I_{n,j}} d_{\tilde{q}_n} u - x \right\|_{C_{\alpha, \beta}} = 0$$

$$= \lim_{n \rightarrow \infty} \left\| \frac{1}{[n]_{q_n}} + \frac{([4]_{q_n} - 2)}{[2]_{q_n}} x \tanh_{q_n}([n]_{q_n} x) + \frac{q_n(q_n^4 - 1)}{[2]_{q_n}^2} [n]_{q_n} x^2 - x \right\|_{C_{\alpha, \beta}} = 0.$$

$$(3) \lim_{n \rightarrow \infty} \|(R_n)_2(u, q_n, \tilde{q}_n; x, y) - y\|_{C_{\alpha, \beta}}$$

$$= \lim_{n \rightarrow \infty} \left\| \frac{[n]_{q_n}}{[2]_{q_n}} \sum_{k=0}^{\infty} a_{n,k}(q_n, x) \int_{I_{n,k}} d_{q_n} t \frac{[n]_{\tilde{q}_n}}{[2]_{\tilde{q}_n}} \times \sum_{j=0}^{\infty} a_{n,j}(\tilde{q}_n, y) \int_{I_{n,j}} u d_{\tilde{q}_n} u - y \right\|_{C_{\alpha, \beta}} = 0$$

$$= \lim_{n \rightarrow \infty} \left\| \frac{1}{[n]_{\tilde{q}_n}} + \frac{([4]_{\tilde{q}_n} - 2)}{[2]_{\tilde{q}_n}} y \tanh_{\tilde{q}_n}([n]_{\tilde{q}_n} y) + \frac{\tilde{q}_n(\tilde{q}_n^4 - 1)}{[2]_{\tilde{q}_n}^2} [n]_{\tilde{q}_n} y^2 - y \right\|_{C_{\alpha, \beta}} = 0$$

$$(4) \lim_{n \rightarrow \infty} \|(R_n)_2(t^2 + u^2, q_n, \tilde{q}_n; x, y) - (x^2 + y^2)\|_{C_{\alpha, \beta}}$$

$$= \lim_{n \rightarrow \infty} \left\| \frac{[n]_{q_n}}{[2]_{q_n}} \sum_{k=0}^{\infty} a_{n,k}(q_n, x) \int_{I_{n,k}} t^2 d_{q_n} t \frac{[n]_{\tilde{q}_n}}{[2]_{\tilde{q}_n}} \right\|_{C_{\alpha, \beta}}$$

$$\begin{aligned}
& \times \sum_{j=0}^{\infty} a_{n,j}(\tilde{q}_n, y) \int_{I_{n,j}} d_{\tilde{q}_n} u \\
& + \frac{[n]_{q_n}}{[2]_{q_n}} \sum_{k=0}^{\infty} a_{n,k}(q_n, x) \int_{I_{n,k}} d_{q_n} t \frac{[n]_{\tilde{q}_n}}{[2]_{\tilde{q}_n}} \\
& \times \sum_{j=0}^{\infty} a_{n,j}(\tilde{q}_n, y) \int_{I_{n,j}} u^2 d_{\tilde{q}_n} u \\
& - (x^2 + y^2) \Big\|_{C_{\alpha,\beta}} \\
& = \lim_{n \rightarrow \infty} \Big\| \left(\frac{[2]_{q_n}^2}{[3]_{q_n} [n]_{q_n}^2} + \frac{3q_n^2}{[n]_{q_n}} x \tanh_{q_n}([n]_{q_n} x) \right. \\
& \quad + \frac{3q_n^5}{[3]_{q_n}} x^2 \\
& \quad + \frac{(q_n - 1)([3] - 2q_n)}{[n]_{q_n}^2} \\
& \quad \times (q_n^3 [n]_{q_n}^3 x^3 \tanh_{q_n}([n]_{q_n} x) \\
& \quad + q_n x^2 [n]_{q_n}^2 (1 \\
& \quad + [2]_{q_n}) + [n]_{q_n} x \tanh_{q_n}([n]_{q_n} x)) \\
& \quad + \frac{[2]_{\tilde{q}_n}^2}{[3]_{\tilde{q}_n} [n]_{\tilde{q}_n}^2} + \frac{3\tilde{q}_n^2}{[n]_{\tilde{q}_n}} y \tanh_{\tilde{q}_n}([n]_{\tilde{q}_n} y) \\
& \quad + \frac{3\tilde{q}_n^5}{[3]_{\tilde{q}_n}} y^2 + \frac{(\tilde{q}_n - 1)([3] - 2\tilde{q}_n)}{[n]_{\tilde{q}_n}^2} \\
& \quad \times (\tilde{q}_n^3 [n]_{\tilde{q}_n}^3 y^3 \tanh_{\tilde{q}_n}([n]_{\tilde{q}_n} y) \\
& \quad + \tilde{q}_n y^2 [n]_{\tilde{q}_n}^2 (1 \\
& \quad + [2]_{\tilde{q}_n}) + [n]_{\tilde{q}_n} y \tanh_{\tilde{q}_n}([n]_{\tilde{q}_n} y)) \Big\|_{C_{\alpha,\beta}} = 0 \quad \blacksquare
\end{aligned}$$

Definition 2.1:

For $m, p \in N^0$ the (m, p) -th order q -moments $(U_{n,m,p})_{\lambda}(q_n, \tilde{q}_n; (x, y))$ for the

operators $(R_n)_{\lambda}(f(t, u), q_n, \tilde{q}_n; (x, y))$ is defined as:

$$(U_{n,m,p})_{\lambda}(x, y) = (R_n)_{\lambda} \left((t - x)_{q_n}^m (u - y)_{\tilde{q}_n}^p, q_n, \tilde{q}_n; (x, y) \right), \lambda = 1, 2.$$

The next lemma gives us the recurrence relations for each $(U_{n,m,p})_{\lambda}$.

Lemma 2.4:

For $m, p \in N^0$ the m -th and p -th order q -moments $(U_{n,m,p})_{\lambda}(x)$ for the operators $(R_n)_{\lambda}(f(t, u), q_n, \tilde{q}_n; (x, y))$ are defined respectively as:

$$\begin{aligned}
(1) & [n]_{q_n} [n]_{\tilde{q}_n} (U_{n,m+1,p+1})_1(q_n, \tilde{q}_n, (x, y)) \\
& = [xD_{q_n} (U_{n,m})_1(q_n, x) - [n]_{q_n} q_n^m x (U_{n,m})_1(q_n, x) \\
& \quad + [n]_{q_n} x \tanh_{q_n}([n]x) (U_{n,m})_1(q_n, q_n x) \\
& \quad + ([m]x \\
& \quad - [n]_{q_n} x^2 \tanh_{q_n}([n]x)(1 \\
& \quad - q_n^m)) (U_{n,m-1})_1(q_n, x)] [yD_{\tilde{q}_n} (U_{n,p})_1(\tilde{q}_n, y) \\
& \quad - [n]_{\tilde{q}_n} \tilde{q}_n^p y (U_{n,p})_1(\tilde{q}_n, y) \\
& \quad + [n]_{\tilde{q}_n} y \tanh_{\tilde{q}_n}([n]y) (U_{n,p})_1(\tilde{q}_n, \tilde{q}_n y) \\
& \quad + ([p]y \\
& \quad - [n]_{\tilde{q}_n} y^2 \tanh_{\tilde{q}_n}([n]y)(1 \\
& \quad - \tilde{q}_n^p)) (U_{n,p-1})_1(\tilde{q}_n, \tilde{q}_n y)], m, p \geq 1 \quad (2.4) \\
(2) & \frac{[n]_{q_n} [n]_{\tilde{q}_n}}{q_n^2 \tilde{q}_n^2} \frac{[m+2]_{q_n} [p+2]_{\tilde{q}_n}}{[m+1]_{q_n} [p+1]_{\tilde{q}_n}}
\end{aligned}$$

$$\begin{aligned}
 & (U_{n,m+1,p+1})_2(q_n, \tilde{q}_n, (x, y)) \\
 &= \left[x D_{q_n} (U_{n,m})_2(q_n, x) \right. \\
 & \quad - \frac{[n]_{q_n}}{q_n^2} \left(q_n^{m+1} x - \frac{[2]_{q_n}}{[n]_{q_n}} \right) (U_{n,m})_2(q_n, x) \\
 & \quad + [n]_{q_n} x \tanh_{q_n}([n]_{q_n} x) (U_{n,m})_2(q_n, q_n x) \\
 & \quad + x (U_{n,m-1})_2(q_n, q_n x) ([m]_{q_n} \\
 & \quad - (1 - q_n^m) [n]_{q_n} x^2 \tanh_{q_n}([n]_{q_n} x)) \\
 & \quad + \frac{[n]^2}{[2]_{q_n} [m+1]_{q_n}} \left(q_n^{m+1} x \right. \\
 & \quad \left. - \frac{1}{q_n^2} \left(q_n^{m+1} x - \frac{[2]_{q_n}}{[n]_{q_n}} \right) \right) (U_{n,m+1})_1(q_n, x) \\
 & \quad \left. - \frac{[n]_{q_n}^2}{[2]_{q_n} [m+1]_{q_n}} \left(\frac{1}{q_n^2} - 1 \right) (U_{n,m+2})_1(q_n, x) \right] \times \\
 & \left[y D_{\tilde{q}_n} (U_{n,p})_2(\tilde{q}_n, y) \right. \\
 & \quad - \frac{[n]_{\tilde{q}_n}}{\tilde{q}_n^2} \left(\tilde{q}_n^{p+1} y - \frac{[2]_{\tilde{q}_n}}{[n]_{\tilde{q}_n}} \right) (U_{n,p})_2(\tilde{q}_n, y) \\
 & \quad + [n]_{\tilde{q}_n} y \tanh_{\tilde{q}_n}([n]_{\tilde{q}_n} y) (U_{n,p})_2(\tilde{q}_n, \tilde{q}_n y) \\
 & \quad + y (U_{n,p-1})_2(\tilde{q}_n, \tilde{q}_n y) ([p]_{\tilde{q}_n} \\
 & \quad - (1 - \tilde{q}_n^p) [n]_{\tilde{q}_n} y^2 \tanh_{\tilde{q}_n}([n]_{\tilde{q}_n} y)) \\
 & \quad + \frac{[n]^2}{[2]_{\tilde{q}_n} [p+1]_{\tilde{q}_n}} \left(\tilde{q}_n^{p+1} y \right. \\
 & \quad \left. - \frac{1}{\tilde{q}_n^2} \left(\tilde{q}_n^{p+1} y - \frac{[2]_{\tilde{q}_n}}{[n]_{\tilde{q}_n}} \right) \right) (U_{n,p+1})_1(\tilde{q}_n, y) \\
 & \quad \left. - \frac{[n]_{\tilde{q}_n}^2}{[2]_{\tilde{q}_n} [p+1]_{\tilde{q}_n}} \left(\frac{1}{\tilde{q}_n^2} \right. \right. \\
 & \quad \left. \left. - 1 \right) (U_{n,p+2})_1(\tilde{q}_n, y) \right], m, p \\
 & \geq 1 \tag{2.5}
 \end{aligned}$$

Proof:

To proof (2.4) and (2.5) using the identities (1.1), (2.1) and lemma 2.3, we have:

For $\lambda = 1$:

$$\begin{aligned}
 & xy D_{q_n, \tilde{q}_n} (U_{n,m,p})_1(q_n, \tilde{q}_n, (x, y)) \\
 &= \left[\sum_{k=0}^{\infty} (t-x)_{q_n}^m D_{q_n} a_{n,k}(q_n, x) \right. \\
 & \quad - [m]_x \sum_{k=0}^{\infty} a_{n,k}(q_n, q_n x) (t \\
 & \quad - x)_{q_n}^{m-1} \left. \sum_{j=0}^{\infty} a_{n,j}(\tilde{q}_n, y) (t-y)_{\tilde{q}_n}^p \right. \\
 & \quad + \left[\sum_{j=0}^{\infty} (t-y)_{\tilde{q}_n}^p D_{\tilde{q}_n} a_{n,k}(\tilde{q}_n, y) \right. \\
 & \quad - [p]_y \sum_{j=0}^{\infty} a_{n,j}(\tilde{q}_n, \tilde{q}_n y) (t \\
 & \quad - y)_{\tilde{q}_n}^{p-1} \left. \sum_{k=0}^{\infty} a_{n,k}(q_n, x) (t-x)_{q_n}^m \right. \\
 & \quad \left. \left[x D_{q_n} (U_{n,m})_1(q_n, x) \right. \right. \\
 & \quad \left. + [m]_x (U_{n,m-1})_1(q_n, q_n x) \right] \left[y D_{\tilde{q}_n} (U_{n,p})_1(\tilde{q}_n, y) \right. \\
 & \quad \left. + [p]_y (U_{n,p-1})_1(\tilde{q}_n, \tilde{q}_n y) \right] \\
 &= \left[\sum_{k=0}^{\infty} ([2k]_{q_n} a_{n,k}(q_n, x) \right. \\
 & \quad - [n]_{q_n} x \tanh_{q_n}([n]_x) a_{n,k}(q_n, q_n x) (t \\
 & \quad - x)_{q_n}^m \left. \sum_{j=0}^{\infty} ([2j]_{\tilde{q}_n} a_{n,j}(\tilde{q}_n, y) \right. \\
 & \quad - [n]_{\tilde{q}_n} y \tanh_{\tilde{q}_n}([n]_y) a_{n,j}(\tilde{q}_n, \tilde{q}_n y) (t \\
 & \quad - y)_{\tilde{q}_n}^p \left. \right].
 \end{aligned}$$

Using the fact: $(t-x)_q^m = (t-x)_q(t-qx)_q^{m-1} = (t-qx)_q^m - x(1-q^m)(t-qx)_q^{m-1}$,

we get:

$$\begin{aligned}
 & \left[x D_{q_n} (U_{n,m})_1(q_n, x) \right. \\
 & \quad \left. + [m]_x (U_{n,m-1})_1(q_n, q_n x) \right] \left[y D_{\tilde{q}_n} (U_{n,p})_1(\tilde{q}_n, y) \right. \\
 & \quad \left. + [p]_y (U_{n,p-1})_1(\tilde{q}_n, \tilde{q}_n y) \right]
 \end{aligned}$$

$$\begin{aligned}
&= \left[[n]_{q_n} \sum_{k=0}^{\infty} a_{n,k}(q_n, x) (t - q_n^m x) (t - x)^m \right. \\
&+ [n]_{q_n} q_n^m x \sum_{k=0}^{\infty} a_{n,k}(q_n, x) (t - x)^m \\
&- [n]_{q_n} x \tanh_{q_n}([n]x) \sum_{k=0}^{\infty} a_{n,k}(q_n, q_n x) (t - q_n x)^m \\
&- q_n x)^m \\
&+ [n]_{q_n} x^2 \tanh_{q_n}([n]x) (1 - q_n^m) \sum_{k=0}^{\infty} a_{n,k}(q_n, q_n x) (t - q_n x)^{m-1} \Big] \times \\
&\left[[n]_{\tilde{q}_n} \sum_{j=0}^{\infty} a_{n,j}(\tilde{q}_n, y) (t - \tilde{q}_n^p y) (t - y)^p \right. \\
&+ [n]_{\tilde{q}_n} \tilde{q}_n^p y \sum_{j=0}^{\infty} a_{n,j}(\tilde{q}_n, y) (t - y)^p \\
&- [n]_{\tilde{q}_n} y \tanh_{\tilde{q}_n}([n]y) \sum_{j=0}^{\infty} a_{n,j}(\tilde{q}_n, \tilde{q}_n y) (t - \tilde{q}_n y)^p \\
&- \tilde{q}_n y)^p \\
&+ [n]_{\tilde{q}_n} y^2 \tanh_{\tilde{q}_n}([n]y) (1 - \tilde{q}_n^p) \sum_{j=0}^{\infty} a_{n,j}(\tilde{q}_n, \tilde{q}_n y) (t - \tilde{q}_n y)^{p-1} \Big]
\end{aligned}$$

Hence,

$$\begin{aligned}
&[n]_{q_n} [n]_{\tilde{q}_n} (U_{n,m+1,p+1})_1(q_n, \tilde{q}_n, (x, y)) \\
&= [xD_{q_n}(U_{n,m})_1(q_n, x) - [n]_{q_n} q_n^m x (U_{n,m})_1(q_n, x) \\
&+ [n]_{q_n} x \tanh_{q_n}([n]x) (U_{n,m})_1(q_n, q_n x) + ([m]x - [n]_{q_n} x^2 \tanh_{q_n}([n]x) (1 - q_n^m)) (U_{n,m-1})_1(q_n, x)] [yD_{\tilde{q}_n}(U_{n,p})_1(\tilde{q}_n, y) \\
&- [n]_{\tilde{q}_n} \tilde{q}_n^p y (U_{n,p})_1(\tilde{q}_n, y) + [n]_{\tilde{q}_n} y \tanh_{\tilde{q}_n}([n]y) (U_{n,p})_1(\tilde{q}_n, \tilde{q}_n y) + ([p]y - [n]_{\tilde{q}_n} y^2 \tanh_{\tilde{q}_n}([n]y) (1 - \tilde{q}_n^p)) (U_{n,p-1})_1(\tilde{q}_n, \tilde{q}_n y)]
\end{aligned}$$

This completes to proof of consequence (2.4);

If $\lambda = 2$, we have:

$$\begin{aligned}
 & D_{q_n, \tilde{q}_n} (U_{n,m,p})_2 (q_n, \tilde{q}_n, (x, y)) \\
 &= \left[\frac{[n]_{q_n}}{[2]_{q_n}} \sum_{k=0}^{\infty} D_{q_n} a_{n,k}(q_n, x) \int_{I_{n,k}} (t \right. \\
 &\quad - x)_{q_n}^m d_{q_n} t \\
 &\quad - [m]_{q_n} \frac{[n]_{q_n}}{[2]_{q_n}} \sum_{k=0}^{\infty} a_{n,k}(q_n, q_n x) \int_{I_{n,k}} (t \\
 &\quad - q_n x)_{q_n}^{m-1} d_{q_n} t \left. \frac{[n]_{\tilde{q}_n}}{[2]_{\tilde{q}_n}} \sum_{j=0}^{\infty} a_{n,j}(\tilde{q}_n, y) \int_{I_{n,j}} (t \right. \\
 &\quad - y)_{\tilde{q}_n}^p d_{\tilde{q}_n} s \\
 &\quad \times \left[\frac{[n]_{\tilde{q}_n}}{[2]_{\tilde{q}_n}} \sum_{j=0}^{\infty} D_{\tilde{q}_n} a_{n,j}(\tilde{q}_n, y) \int_{I_{n,k}} (t \right. \\
 &\quad - x)_{\tilde{q}_n}^p d_{\tilde{q}_n} s \\
 &\quad - [p]_{\tilde{q}_n} \frac{[n]_{\tilde{q}_n}}{[2]_{\tilde{q}_n}} \sum_{j=0}^{\infty} a_{n,j}(\tilde{q}_n, \tilde{q}_n y) \int_{I_{n,k}} (t \\
 &\quad - \tilde{q}_n y)_{\tilde{q}_n}^{p-1} d_{\tilde{q}_n} s \left. \frac{[n]_{q_n}}{[2]_{q_n}} \sum_{k=0}^{\infty} a_{n,k}(q_n, x) \int_{I_{n,k}} (t \right. \\
 &\quad - x)_{q_n}^m d_{q_n} t \\
 &\quad \times [xD_{q_n} (U_{n,m})_2 (q_n, x) \\
 &\quad + [m]x (U_{n,m-1})_2 (q_n, q_n x)] [yD_{\tilde{q}_n} (U_{n,p})_2 (\tilde{q}_n, y) \\
 &\quad + [p]y (U_{n,p-1})_2 (\tilde{q}_n, \tilde{q}_n y)] \\
 &= \left[\frac{[n]_{q_n}}{[2]_{q_n}} \sum_{k=0}^{\infty} xD_{q_n} a_{n,k}(q_n, x) \int_{I_{n,k}} (t \right. \\
 &\quad - x)_{q_n}^m d_{q_n} t \left. \frac{[n]_{\tilde{q}_n}}{[2]_{\tilde{q}_n}} \sum_{j=0}^{\infty} yD_{\tilde{q}_n} a_{n,j}(\tilde{q}_n, y) \int_{I_{n,j}} (t \right. \\
 &\quad - y)_{\tilde{q}_n}^p d_{\tilde{q}_n} s \left. \right] \\
 &= \left[\frac{[n]_{q_n}}{[2]_{q_n}} \sum_{k=0}^{\infty} [2k]_{q_n} a_{n,k}(q_n, x) \int_{I_{n,k}} (t \right. \\
 &\quad - x)_{q_n}^m d_{q_n} t \\
 &\quad - \frac{[n]_{q_n}^2}{[2]_{q_n}} x \tanh_{q_n}([n]_{q_n} x) \sum_{k=0}^{\infty} a_{n,k}(q_n, q_n x) \int_{I_{n,k}} (t \\
 &\quad - x)_{q_n}^m d_{q_n} t \left. \frac{[n]_{\tilde{q}_n}}{[2]_{\tilde{q}_n}} \sum_{j=0}^{\infty} [2j]_{\tilde{q}_n} a_{n,j}(\tilde{q}_n, y) \int_{I_{n,j}} (t \right. \\
 &\quad - y)_{\tilde{q}_n}^p d_{\tilde{q}_n} s \\
 &\quad - \frac{[n]_{\tilde{q}_n}^2}{[2]_{\tilde{q}_n}} y \tanh_{\tilde{q}_n}([n]_{\tilde{q}_n} y) \sum_{j=0}^{\infty} a_{n,j}(\tilde{q}_n, \tilde{q}_n y) \int_{I_{n,j}} (t \\
 &\quad - y)_{\tilde{q}_n}^p d_{\tilde{q}_n} s \left. \right] \\
 &= \left[\frac{[n]_{q_n}^2}{q_n^2 [2]_{q_n}} \frac{1}{[m+1]_{q_n}} \sum_{k=0}^{\infty} a_{n,k}(q_n, x) \left(\frac{[2k+2]_{q_n}}{[n]_{q_n}} \right. \right. \\
 &\quad - x \left. \right)^{m+1} \left(\frac{q_n^2 [2k]_{q_n} + [2]_{q_n} - [2]_{q_n}}{[n]_{q_n}} \right. \\
 &\quad \left. \left. + q_n^{m+1} x - q_n^{m+1} x \right) \right. \\
 &\quad - \frac{[n]_{q_n}^2}{[2]_{q_n}} \frac{1}{[m+1]_{q_n}} \sum_{k=0}^{\infty} a_{n,k}(q_n, x) \left(\frac{[2k]_{q_n}}{[n]_{q_n}} \right. \\
 &\quad - x \left. \right)^{m+1} \left(\frac{[2k]_{q_n}}{[n]_{q_n}} + q_n^{m+1} x - q_n^{m+1} x \right) \\
 &\quad + \frac{[n]_{q_n}^2}{[2]_{q_n}} x \tanh_{q_n}([n]_{q_n} x) \sum_{k=0}^{\infty} a_{n,k}(q_n, q_n x) \int_{I_{n,k}} (t \\
 &\quad - q_n x)_{q_n}^m d_{q_n} t \\
 &\quad + \frac{[n]_{q_n}^2}{[2]_{q_n}} x^2 \tanh_{q_n}([n]_{q_n} x) (1 \\
 &\quad - q_n^m) \sum_{k=0}^{\infty} a_{n,k}(q_n, q_n x) \int_{I_{n,k}} (t \\
 &\quad - q_n x)_{q_n}^{m-1} d_{q_n} t \left. \right] \times
 \end{aligned}$$

$$\begin{aligned}
& \left[\frac{[n]_{\tilde{q}_n}^2}{\tilde{q}_n^2 [2]_{\tilde{q}_n}} \frac{1}{[p+1]_{\tilde{q}_n}} \sum_{j=0}^{\infty} a_{n,j}(\tilde{q}_n, y) \left(\frac{[2j+2]_{q_n}}{[n]_{\tilde{q}_n}} \right. \right. \\
& \left. \left. - y \right)^{p+1} \left(\frac{\tilde{q}_n^2 [2j]_{\tilde{q}_n} + [2]_{\tilde{q}_n} - [2]_{\tilde{q}_n}}{[n]_{\tilde{q}_n}} + \tilde{q}_n^{p+1} y \right. \right. \\
& \left. \left. - \tilde{q}_n^{p+1} y \right) \right. \\
& \left. - \frac{[n]_{\tilde{q}_n}^2}{[2]_{\tilde{q}_n}} \frac{1}{[p+1]_{\tilde{q}_n}} \sum_{j=0}^{\infty} a_{n,j}(\tilde{q}_n, y) \left(\frac{[2j]_{\tilde{q}_n}}{[n]_{\tilde{q}_n}} \right. \right. \\
& \left. \left. - y \right)^{p+1} \left(\frac{[2j]_{\tilde{q}_n}}{[n]_{\tilde{q}_n}} + \tilde{q}_n^{p+1} y - \tilde{q}_n^{p+1} y \right) \right. \\
& \left. + \frac{[n]_{\tilde{q}_n}^2}{[2]_{\tilde{q}_n}} y \tanh_{\tilde{q}_n}([n]_{\tilde{q}_n} y) \sum_{j=0}^{\infty} a_{n,j}(\tilde{q}_n, \tilde{q}_n y) \int_{I_{n,j}} (t \right. \\
& \left. - \tilde{q}_n y)_{\tilde{q}_n}^p d_{\tilde{q}_n} s \right. \\
& \left. + \frac{[n]_{\tilde{q}_n}^2}{[2]_{\tilde{q}_n}} y^2 \tanh_{\tilde{q}_n}([n]_{\tilde{q}_n} y) (1 \right. \\
& \left. - \tilde{q}_n^p) \sum_{j=0}^{\infty} a_{n,j}(\tilde{q}_n, \tilde{q}_n y) \int_{I_{n,j}} (t \right. \\
& \left. - \tilde{q}_n y)_{\tilde{q}_n}^{p-1} d_{\tilde{q}_n} s \right] \\
& = \left[\frac{[n]_{q_n} [m+2]_{q_n}}{q_n^2 [m+1]_{q_n}} (U_{n,m+1})_2(x) \right. \\
& + \frac{[n]_{q_n}^2}{[2]_{q_n} [m+1]_{q_n}} \left(\frac{1}{q_n^2} - 1 \right) (U_{n,m+2})_1(x) \\
& + \frac{[n]_{q_n}}{q_n^2} \left(q_n^{m+1} x - \frac{[2]_{q_n}}{[n]_{q_n}} \right) (U_{n,m})_2(x) \\
& + \frac{[n]_{q_n}^2}{q_n^2 [2]_{q_n}} \left(q_n^{m+1} x - \frac{[2]_{q_n}}{[n]_{q_n}} \right) (U_{n,m+1})_1(x) \\
& - \frac{[n]^2}{[2]_{q_n} [m+1]_{q_n}} q_n^{m+1} x (U_{n,m+1})_1(x) \\
& - [n]_{q_n} x \tanh_{q_n}([n]_{q_n} x) (U_{n,m})_2(q_n x) \\
& + [n]_{q_n} x^2 \tanh_{q_n}([n]_{q_n} x) \\
& \left. \times (1 - q_n^m) (U_{n,m-1})_2(q_n x) \right] \\
& \times \left[\frac{[n]_{\tilde{q}_n} [p+2]_{\tilde{q}_n}}{\tilde{q}_n^2 [p+1]_{\tilde{q}_n}} (U_{n,p+1})_2(y) \right. \\
& + \frac{[n]_{\tilde{q}_n}^2}{[2]_{\tilde{q}_n} [p+1]_{\tilde{q}_n}} \left(\frac{1}{\tilde{q}_n^2} - 1 \right) (U_{n,p+2})_1(y) \\
& + \frac{[n]_{\tilde{q}_n}}{\tilde{q}_n^2} \left(\tilde{q}_n^{p+1} y - \frac{[2]_{\tilde{q}_n}}{[n]_{\tilde{q}_n}} \right) (U_{n,p})_2(y) \\
& + \frac{[n]_{\tilde{q}_n}^2}{\tilde{q}_n^2 [2]_{\tilde{q}_n}} \left(\tilde{q}_n^{p+1} y - \frac{[2]_{\tilde{q}_n}}{[n]_{\tilde{q}_n}} \right) (U_{n,p+1})_1(y) \\
& - \frac{[n]^2}{[2]_{\tilde{q}_n} [p+1]_{\tilde{q}_n}} \tilde{q}_n^{p+1} y (U_{n,p+1})_1(y) \\
& - [n]_{\tilde{q}_n} y \tanh_{\tilde{q}_n}([n]_{\tilde{q}_n} y) (U_{n,p})_2(\tilde{q}_n y) \\
& + [n]_{\tilde{q}_n} y^2 \tanh_{\tilde{q}_n}([n]_{\tilde{q}_n} y) \\
& \left. \times (1 - \tilde{q}_n^p) (U_{n,p-1})_2(\tilde{q}_n y) \right]
\end{aligned}$$

Hence:

$$\begin{aligned}
 & \frac{[n]_{q_n}[n]_{\tilde{q}_n}[m+2]_{q_n}[p+2]_{\tilde{q}_n}}{q_n^2\tilde{q}_n^2} \times \\
 & (U_{n,m+1,p+1})_2(q_n, \tilde{q}_n, (x, y)) \\
 & = \left[xD_{q_n}(U_{n,m})_2(q_n, x) \right. \\
 & - \frac{[n]_{q_n}}{q_n^2} \left(q_n^{m+1}x - \frac{[2]_{q_n}}{[n]_{q_n}} \right) (U_{n,m})_2(q_n, x) \\
 & + [n]_{q_n}x \tanh_{q_n}([n]_{q_n}x) (U_{n,m})_2(q_n, q_nx) \\
 & + x(U_{n,m-1})_2(q_n, q_nx) ([m]_{q_n} \\
 & - (1 - q_n^m)[n]_{q_n}x^2 \tanh_{q_n}([n]_{q_n}x)) \\
 & + \frac{[n]^2}{[2]_{q_n}[m+1]_{q_n}} \left(q_n^{m+1}x \right. \\
 & - \frac{1}{q_n^2} \left(q_n^{m+1}x - \frac{[2]_{q_n}}{[n]_{q_n}} \right) \left. \right) (U_{n,m+1})_1(q_n, x) \\
 & - \frac{[n]_{q_n}^2}{[2]_{q_n}[m+1]_{q_n}} \left(\frac{1}{q_n^2} \right. \\
 & - 1 \left. \right) (U_{n,m+2})_1(q_n, x) \left. \right] \left[yD_{\tilde{q}_n}(U_{n,p})_2(\tilde{q}_n, y) \right. \\
 & - \frac{[n]_{\tilde{q}_n}}{\tilde{q}_n^2} \left(\tilde{q}_n^{p+1}y - \frac{[2]_{\tilde{q}_n}}{[n]_{\tilde{q}_n}} \right) (U_{n,p})_2(\tilde{q}_n, y) \\
 & + [n]_{\tilde{q}_n}y \tanh_{\tilde{q}_n}([n]_{\tilde{q}_n}y) (U_{n,p})_2(\tilde{q}_n, \tilde{q}_ny) \\
 & + y(U_{n,p-1})_2(\tilde{q}_n, \tilde{q}_ny) ([p]_{\tilde{q}_n} \\
 & - (1 - \tilde{q}_n^p)[n]_{\tilde{q}_n}y^2 \tanh_{\tilde{q}_n}([n]_{\tilde{q}_n}y)) \\
 & + \frac{[n]^2}{[2]_{\tilde{q}_n}[p+1]_{\tilde{q}_n}} \left(\tilde{q}_n^{p+1}y \right. \\
 & - \frac{1}{\tilde{q}_n^2} \left(\tilde{q}_n^{p+1}y - \frac{[2]_{\tilde{q}_n}}{[n]_{\tilde{q}_n}} \right) \left. \right) (U_{n,p+1})_1(\tilde{q}_n, y) \\
 & - \frac{[n]_{\tilde{q}_n}^2}{[2]_{\tilde{q}_n}[p+1]_{\tilde{q}_n}} \left(\frac{1}{\tilde{q}_n^2} \right. \\
 & - 1 \left. \right) (U_{n,p+2})_1(\tilde{q}_n, y) \left. \right]. \quad \blacksquare
 \end{aligned}$$

3. Main Results

The next theorem is a Voronovaskaja-type asymptotic formula for $(R_n)_\lambda(f(t, s), q_n, \tilde{q}_n; (x, y))$, $\lambda = 1, 2$.

Theorem 3.1:

For $f \in C_{\alpha_1, \alpha_2}$, suppose that $\frac{\partial^2 f(x, y)}{\partial^2 x}$, $\frac{\partial^2 f(x, y)}{\partial^2 y}$ and $\frac{\partial^2 f(x, y)}{\partial x \partial y}$ exist and are continuous at a point $(x, y) \in ([0, \infty) \times [0, \infty))$, then :

$$\begin{aligned}
 & \lim_{n \rightarrow \infty} n[(R_n)_\lambda(f(t, s), q_n, \tilde{q}_n; (x, y)) \\
 & - f(x, y)] \\
 & = \begin{cases} \frac{1}{2} (xf''_{xx}(x, y) + yf''_{yy}(x, y)), & \text{if } \lambda = 1 \\ f'_x(x, y) + f'_y(x, y) \\ + \frac{1}{2} \left(xf''_{xx}(x, y) + yf''_{yy}(x, y) \right) + f''_{x,y}(x, y), & \text{if } \lambda = 2. \end{cases}
 \end{aligned}$$

Proof:

if $\lambda = 1$, we get:

By Taylor's formula for f , about the point (x, y) , we have: V.Kac and P.Cheung, (2002)

$$\begin{aligned}
 & f(t, s) \\
 & = f(x, y) + f'_x(x, y)(t - x) \\
 & + f'_y(x, y)(u - y) \\
 & + \frac{1}{2} \{ f''_{xx}(x, y)(t - x)^2 + 2f''_{xy}(t - x)(u - y) \\
 & + f''_{yy}(x, y)(u - y)^2 \} \\
 & + \varphi_1((t, s); x, y) \sqrt{(t - x)^4 + (u - y)^4},
 \end{aligned}$$

where $\varphi_1(t, s) = \varphi_1((t, s); (x, y))$ is a function from the space $C_{C_{\alpha_1, \alpha_2}}([0, \infty) \times [0, \infty))$, and $\varphi_1(t, s) \rightarrow (0, 0)$ as $(t, s) \rightarrow (x, y)$ thus, $\varphi_1(x, y) = 0$.

$$\begin{aligned}
& (R_n)_\lambda(f(t, s), q_n, \tilde{q}_n; (x, y)) \\
&= f(x, y) \\
&+ f'_x(x, y)(R_n)_\lambda((t - x), q_n, \tilde{q}_n; (x, y)) \\
&+ f'_y(x, y)(R_n)_\lambda((s - y), q_n, \tilde{q}_n; (x, y)) \\
&+ \frac{1}{2} f''_{xx}(x, y)(R_n)_\lambda((t - x)^2, q_n, \tilde{q}_n; (x, y)) \\
&+ f''_{xy}(x, y)(R_n)_\lambda((t - x)(s - y), q_n, \tilde{q}_n; (x, y)) \\
&+ f''_{yy}(x, y)(R_n)_\lambda((s - y)^2, q_n, \tilde{q}_n; (x, y)). \\
&+ (R_n)_\lambda(\varphi_1(t, s) \sqrt{(t - x)^4 + (s - y)^4}, \\
&q_n, \tilde{q}_n; (x, y)) \quad (3.1)
\end{aligned}$$

$$= I_1 + I_2$$

Using lemma 2.5, we have:

For $\lambda = 1$

$$\begin{aligned}
I_1 &= f(x, y) + f'_x(x, y)[x \tanh_{q_n}([n]x) - x] \\
&+ f'_y(x, y)[y \tanh_{\tilde{q}_n}([n]y) - y] \\
&+ \frac{1}{2} f''_{xx}(x, y) \left[\frac{x}{[n]_{q_n}} \tanh_{q_n}([n]x) \right. \\
&\quad \left. - [2]_{q_n} x^2 \tanh_{q_n}([n]x) + 2q_n x^2 \right] \\
&+ f''_{xy}(x, y) [x \tanh_{q_n}([n]x) \\
&\quad - x][y \tanh_{\tilde{q}_n}([n]y) - y] \\
&+ \frac{1}{2} f''_{yy}(x, y) \left[\frac{y}{[n]_{\tilde{q}_n}} \tanh_{\tilde{q}_n}([n]y) \right. \\
&\quad \left. - [2]_{\tilde{q}_n} y^2 \tanh_{\tilde{q}_n}([n]y) + 2\tilde{q}_n y^2 \right]
\end{aligned}$$

Then

$$\lim_{n \rightarrow \infty} [n] I_1 = \frac{1}{2} (x f''_{xx}(x, y) + y f''_{yy}(x, y))$$

Now for $\lambda = 2$ we have:

$$\begin{aligned}
I_1 &= f(x, y) [(q_n - 1)[n]_{q_n} x \tanh_{q_n}([n]_{q_n} x) \\
&\quad + 1] + f'_x(x, y) \left[\frac{1}{[n]_{q_n}} \right. \\
&\quad \left. + x \left(\frac{([4]_{q_n} - 2)}{[2]_{q_n}} \tanh_{q_n}([n]_{q_n} x) - 1 \right) \right. \\
&\quad \left. + (q_n - 1)[n]_{q_n} x^2 \left(\frac{(q_n^2 + 1)q_n}{[2]_{q_n}} \right. \right. \\
&\quad \left. \left. - \tanh_{q_n}([n]_{q_n} x) \right) \right] \\
&+ f'_y(x, y) \left[\frac{1}{[n]_{\tilde{q}_n}} \right. \\
&\quad \left. + y \left(\frac{([4]_{\tilde{q}_n} - 2)}{[2]_{\tilde{q}_n}} \tanh_{\tilde{q}_n}([n]_{\tilde{q}_n} y) - 1 \right) \right. \\
&\quad \left. + (\tilde{q}_n - 1)[n]_{\tilde{q}_n} y^2 \left(\frac{(\tilde{q}_n^2 + 1)\tilde{q}_n}{[2]_{\tilde{q}_n}} \right. \right. \\
&\quad \left. \left. - \tanh_{\tilde{q}_n}([n]_{\tilde{q}_n} y) \right) \right] \\
&+ \frac{1}{2} f''_{xx}(x, y) \left[\frac{1}{[n]_{q_n}} x (([4]_{q_n} + q_n \right. \\
&\quad \left. - 2) \tanh_{q_n}([n]_{q_n} x) - [2]_{q_n} \right. \\
&\quad \left. + x^2 \left(\frac{3q_n^5}{[3]_{q_n}} \right. \right. \\
&\quad \left. \left. + q_n^5 - q_n(q_n - 1)(2q_n - 1) - ([4]_{q_n} \right. \right. \\
&\quad \left. \left. - 2) \tanh_{q_n}([n]_{q_n} x) \right) \right. \\
&\quad \left. + q_n(q_n \right. \\
&\quad \left. - 1)[n]_{q_n} x^3 \left((q_n^2([3]_{q_n} - 2q_n) \right. \right. \\
&\quad \left. \left. + 1) \tanh_{q_n}([n]x) + ([3]_{q_n} - q_n) \right) \right. \\
&\quad \left. + \frac{[2]_{q_n}^2}{[3]_{q_n} [n]_{q_n}^2} \right]
\end{aligned}$$

$$\begin{aligned}
 & + f''_{xy}(x, y) \left[\frac{1}{[n]_{q_n}} \right. \\
 & + x \left(\frac{([4]_{q_n} - 2)}{[2]_{q_n}} \tanh_{q_n}([n]_{q_n} x) \right. \\
 & \left. \left. - 1 \right) \right. \\
 & + (q_n - 1)[n]_{q_n} x^2 \left(\frac{(q_n^2 + 1)q_n}{[2]_{q_n}} \right. \\
 & \left. \left. - \tanh_{q_n}([n]_{q_n} x) \right) \right] \left[\frac{1}{[n]_{\tilde{q}_n}} \right. \\
 & + y \left(\frac{([4]_{\tilde{q}_n} - 2)}{[2]_{\tilde{q}_n}} \tanh_{\tilde{q}_n}([n]_{\tilde{q}_n} y) \right. \\
 & \left. \left. - 1 \right) \right. \\
 & + (\tilde{q}_n - 1)[n]_{\tilde{q}_n} y^2 \left(\frac{(\tilde{q}_n^2 + 1)\tilde{q}_n}{[2]_{\tilde{q}_n}} \right. \\
 & \left. \left. - \tanh_{\tilde{q}_n}([n]_{\tilde{q}_n} y) \right) \right] \\
 & + \frac{1}{2} f''_{yy}(x, y) \left[\frac{1}{[n]_{\tilde{q}_n}} y \left(([4]_{\tilde{q}_n} + \tilde{q}_n \right. \right. \\
 & \left. \left. - 2) \tanh_{\tilde{q}_n}([n]_{\tilde{q}_n} y) - [2]_{\tilde{q}_n} \right) \right. \\
 & + y^2 \left(\frac{3\tilde{q}_n^5}{[3]_{\tilde{q}_n}} \right. \\
 & + \tilde{q}_n^5 - \tilde{q}_n(\tilde{q}_n - 1)(2\tilde{q}_n - 1) \\
 & \left. - ([4]_{\tilde{q}_n} - 2) \tanh_{\tilde{q}_n}([n]_{\tilde{q}_n} y) \right) \\
 & + \tilde{q}_n(\tilde{q}_n \\
 & - 1)[n]_{\tilde{q}_n} y^3 \left((\tilde{q}_n^2([3]_{\tilde{q}_n} - 2\tilde{q}_n) \right. \\
 & + 1) \tanh_{\tilde{q}_n}([n]_{\tilde{q}_n} y) + ([3]_{\tilde{q}_n} \\
 & \left. - \tilde{q}_n) \right) + \frac{[2]_{\tilde{q}_n}^2}{[3]_{\tilde{q}_n}[n]_{\tilde{q}_n}^2} \left. \right]
 \end{aligned}$$

Then, $\lim_{n \rightarrow \infty} [n] I_1 = f'_x(x, y) + f'_y(x, y) + \frac{1}{2} (x f''_{xx}(x, y) + y f''_{yy}(x, y)) + f''_{xy}(x, y)$

$$\begin{aligned}
 I_2 &= \lim_{n \rightarrow \infty} [n] (R_n)_\lambda \\
 & \left(\varphi_1(t, s) \sqrt{(t-x)^4 + (u-y)^4}, q_n, \tilde{q}_n; (x, y) \right)
 \end{aligned}$$

Now, we must show that the term $[n] (R_n)_\lambda (\varphi_1(t, s) \sqrt{(t-x)^4 + (u-y)^4}, q_n, \tilde{q}_n; (x, y))$ as $n \rightarrow \infty$.

By Cauchy-Schwartz inequality, we get:

$$\begin{aligned}
 & [n] \left| (R_n)_\lambda \left(\varphi_1(t, s) \sqrt{(t-x)^4 + (u-y)^4}, q_n, \tilde{q}_n; (x, y) \right) \right| \\
 & \leq [n] [(R_n)_\lambda (\varphi_1^2(t, s), q_n, \tilde{q}_n; x, y)]^{1/2} [(R_n)_1((t-x)^4, q_n; x) + (R_n)_\lambda((u-y)^4, \tilde{q}_n; y)]^{1/2}
 \end{aligned}$$

Now, by the properties of $\varphi_1(x, y) = 0$ as $t \rightarrow x$ and $s \rightarrow y$ we get

$$[(R_n)_\lambda(1, q_n, \tilde{q}_n; x, y)] = \varphi_1^2(x, y) = 0$$

$$\begin{aligned}
 & \lim_{n \rightarrow \infty} (R_n)_\lambda(\varphi_1^2(t, u), q_n, \tilde{q}_n; x, y) \\
 & = \varphi_1^2(x, y) = 0.
 \end{aligned}$$

Thus, from lemma 2.3, we get:

$$\begin{aligned}
 & \lim_{n \rightarrow \infty} [(R_n)_\lambda((t-x)^4, q_n; x) \\
 & + (S_n)_1((u-y)^4, \tilde{q}_n; y)]^{1/2} \\
 & = \lim_{n \rightarrow \infty} [(R_n)_\lambda((t-x)^4, q_n; x) \\
 & + (R_n)_\lambda((u-y)^4, \tilde{q}_n; y)]^{1/2} \\
 & \rightarrow 0 \text{ as } n \rightarrow \infty.
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 & \lim_{n \rightarrow \infty} [n] (R_n)_\lambda \left(\varphi_1(t, s) \sqrt{(t-x)^4 + (u-y)^4}; x, y \right) \\
 & \rightarrow 0 \text{ as } n \rightarrow \infty.
 \end{aligned}$$

Hence, the proof is complete ■

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مبرهنة فرونوفسكيا من النمط q لمؤثرات زاز-ماركجان

في فضاء البعدين

هديل عمر مسلم الخالد

الجامعة التقنية الجنوبية-الكلية التقنية الهندسية-بصرة

المستخلص:

في هذا البحث عرفنا متتابعة من المؤثرات $Sz\ddot{a}sz-Mirakjan$ من النمط q في فضاء البعدين $C_{\alpha,\beta}([0, \infty) \times [0, \infty))$ ورمزنا لهم بالرمز $(R_n)_\lambda(f(t,s); x, y), \lambda = 1, 2$ ، بداية برهنا تقارب هذه المؤثرات إلى الدالة $f(x, y)$ عندما n تقترب إلى اللانهاية ثم عرفنا العزم ووجدنا الصيغة التكرارية لكل من المؤثرات اعلاه. وأخيرا اثبتنا صيغة فرونوفسكيا للتقارب للمؤثرين.